Math 213 First In-class Exam
Version B
Please show your computations, not only your answers.

Room B239, 9:55am - 10:45am, February 16, 2005 **NAME:**
Márton Balázs **Dis. Session:**

1. The main cable of a suspension bridge follows the curve

$$y = 0.04 \cdot (e^{2x} + e^{-2x})$$

(where both x and y are measured in thousand feet) on the interval $-1 \le x \le 1$.

(a) (10 points) Compute the area enclosed between the main cable, the surface of the bridge at y=0, and the two vertical pillars at $x = -1$ and $x = 1$.

Solution: The area is given by

$$\begin{aligned}\int_{-1}^{1} 0.04 \cdot (e^{2x} + e^{-2x})\,dx &= 0.04 \cdot \left[\frac{e^{2x}}{2} + \frac{e^{-2x}}{-2}\right]_{-1}^{1}\\ &= 0.04 \cdot \left(\frac{e^{2}}{2} + \frac{e^{-2}}{-2}\right) - 0.04 \cdot \left(\frac{e^{-2}}{2} + \frac{e^{2}}{-2}\right)\\ &= 0.04\left(e^2 - e^{-2}\right) \simeq 0.29 \text{ (million feet}^2).\end{aligned}$$

(b) (**Bonus question**, only try when all other problems are completed and checked, additional 5 points) Smaller vertical cables transmit support from the main cable to the surface of the bridge. These cables are placed at every five feet (or every 0.005 thousand feet (!)) between the two pillars. Based on part (a), give an approximation of the total length of these vertical cables on the bridge.

Solution: The area computed in part (a) is approximated by little rectangles having the vertical cables as sides, and $\Delta x = 0.005$ as base. To be more precise, using the trapezoidal rule with $n = 400$ intervals from -1 to 1, so as to $\Delta x = 0.005$:

$$\begin{aligned}0.04\left(e^2 - e^{-2}\right) &= \int_{-1}^{1} 0.04 \cdot (e^{2x} + e^{-2x})\,dx\\ &\simeq \frac{1-(-1)}{400}\left[\sum_{i=1}^{399} 0.04 \cdot (e^{2x_i} + e^{-2x_i}) + \frac{1}{2} \cdot 0.04 \cdot (e^{2x_0} + e^{-2x_0}) + \frac{1}{2} \cdot 0.04 \cdot (e^{2x_{400}} + e^{-2x_{400}})\right]\\ &= \frac{1}{200}\left[\sum_{i=1}^{399} 0.04 \cdot (e^{2x_i} + e^{-2x_i}) + \frac{1}{2} \cdot 0.04 \cdot (e^{-2} + e^{2}) + \frac{1}{2} \cdot 0.04 \cdot (e^{2} + e^{-2})\right]\\ &= \frac{1}{200}\left[\sum_{i=1}^{399} 0.04 \cdot (e^{2x_i} + e^{-2x_i}) + 0.04 \cdot (e^{2} + e^{-2})\right].\end{aligned}$$

The total length of the vertical cables is given by the sum seen in this display. Expressing it from the left and right hand-sides, our answer is

$$\sum_{i=1}^{399} 0.04 \cdot (e^{2x_i} + e^{-2x_i}) = 0.04 \cdot 200(e^2 - e^{-2}) - 0.04 \cdot (e^2 + e^{-2}) = 7.96e^2 - 8.04e^{-2}$$

$$\simeq 57.7 \text{ (thousand feet)} \simeq 10.9 \text{ (miles)}.$$

To take these cables into account on both sides of the bridge, we multiply the result by two and get a total length of 115 thousand feet or 21.9 miles.

2. (15 points) Starting from the age of 20, Mr. Jones earned money at a rate of $[t+20]$ (thousand dollars/year) when he was t years old. He also had expenses from the age of 20, he spent money at a rate of $[1.4 \cdot t + 2]$ (thousand dollars/year). He has just turned 60. How much money has he saved until now? (Assume he didn't invest in any kind of investment or bank account.)

Solution: The rate of savings is given by salary − expenses, i.e. $t + 20 - (1.4t + 2) = 18 - 0.4t$, regardless of which is larger. The total savings are therefore given by

$$\int_{20}^{60} (18 - 0.4t)\, dt = [18t - 0.2t^2]\Big|_{20}^{60} = 80 \text{ (thousand dollars)}.$$

3. Find the area under $f(x) = 4x^2$, on the interval $-1 \leq x \leq 1$,

(a) (5 points) using the trapezoidal rule with $n = 4$ intervals,

Solution: By the trapezoidal rule, we compute

$$\int_{-1}^{1} 4x^2\, dx \simeq \frac{1-(-1)}{4} \left[\frac{1}{2} \cdot 4 \cdot (-1)^2 + 4 \cdot (-0.5)^2 + 4 \cdot 0^2 + 4 \cdot 0.5^2 + \frac{1}{2} \cdot 4 \cdot 1^2\right] = 3.$$

(b) (5 points) using Simpson's rule:

$$\int_a^b f(x)\, dx$$

$$\simeq \frac{b-a}{3n} \cdot [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

with $n = 4$ intervals,

Solution: By Simpson's rule, we compute

$$\int_{-1}^{1} 4x^2\, dx \simeq \frac{1-(-1)}{3 \cdot 4} \left[4 \cdot (-1)^2 + 4 \cdot 4 \cdot (-0.5)^2 + 2 \cdot 4 \cdot 0^2 + 4 \cdot 4 \cdot 0.5^2 + 4 \cdot 1^2\right] = \frac{8}{3}.$$

(c) (3 points) by exact integration.

Solution:

$$\int_{-1}^{1} 4x^2\,dx = 4\frac{x^3}{3}\Big|_{-1}^{1} = \frac{4}{3} - \frac{-4}{3} = \frac{8}{3}.$$

(d) (2 points) Compare the accuracy of the results. Can you explain the accuracy of Simpson's rule in this case?

Solution: Simpson's rule is completely accurate, while the trapezoidal rule overestimates the area. Simpson's rule is exact in this case since this method works with parabolas, which can of course precisely be fitted to the parabola $4x^2$ given in the problem.

4. (10 points) Compute

$$\int_{0}^{2} (6x+9)e^{x^2+3x}\,dx.$$

Solution: Substitute $u = x^2 + 3x$, then $du = (2x+3)\,dx$. Therefore

$$\int_{0}^{2} (6x+9)e^{x^2+3x}\,dx = 3\int_{0}^{2} e^{x^2+3x}\,[(2x+3)\,dx] = 3\int_{0}^{10} e^{u}\,du = 3e^{u}\Big|_{0}^{10} = 3e^{10} - 3 \simeq 66\,076.$$

5. (10 points) Compute

$$\int_{0}^{2} (6x+9)e^{3x}\,dx.$$

Solution: Integrate by parts with $u = 6x+9$, $v = e^{3x}/3$:

$$\int_{0}^{2} (6x+9)e^{3x}\,dx = (6x+9)\cdot\frac{e^{3x}}{3}\Big|_{0}^{2} - \int_{0}^{2} 6\,\frac{e^{3x}}{3}\,dx = (6x+9)\cdot\frac{e^{3x}}{3}\Big|_{0}^{2} - 6\,\frac{e^{3x}}{9}\Big|_{0}^{2}$$

$$= (6\cdot 2+9)\cdot\frac{e^{6}}{3} - (0+9)\cdot\frac{1}{3} - 2\frac{e^{6}}{3} + \frac{2}{3} = \frac{19}{3}e^{6} - \frac{7}{3} \simeq 2552.7.$$