Math 213 Second In-class Exam
Version A
Please show your computations, not only your answers.

Room B239, 9:55am - 10:45am, March 18, 2005 **NAME:**
Márton Balázs **Dis. Session:**

1. (15 points) What is the minimum length of a fence **used on three** out of the four **sides** of a rectangular property having an area of 1000 square yards?

Solution: Let the two parallel edges of the property with fences have each length x, and the remaining edge with a fence have length y. Then we have to minimize $f(x, y) = 2x + y$ subject to the constraint $xy = 1000$, or $g(x, y) = xy - 1000 = 0$. Consider the Lagrange function and its first-order derivatives:

$$\begin{aligned} F(x, y, \lambda) &= 2x + y - \lambda(xy - 1000), \\ F_x(x, y, \lambda) &= 2 - \lambda y, \\ F_y(x, y, \lambda) &= 1 - \lambda x, \\ F_\lambda(x, y, \lambda) &= -xy + 1000. \end{aligned}$$

The length of the fence must have a minimum, and the corresponding (x, y) pair is found among the solutions of the system obtained by making the last three equations zero. From $F_x = 0$, we conclude $\lambda = 2/y$, from $F_y = 0$ we derive $\lambda = 1/x$, therefore we obtain $2/y = 1/x$, or $y = 2x$. Plugging this back to $F_\lambda = 0$, we have

$$\begin{aligned} -x \cdot 2x + 1000 &= 0 \\ 2x^2 &= 1000 \\ x &= \sqrt{500}. \end{aligned}$$

Hence $y = 2x = 2\sqrt{500}$, and the minimum length of the fence is

$$f(\sqrt{500}, 2\sqrt{500}) = 2\sqrt{500} + 2\sqrt{500} \simeq 89.4 \text{ (yards)}.$$

2.(a) (10 points) Find the capital value (i.e. the present value up to an infinite time) of a continuous money flow flowing with rate $f(x) = 12\,000 \cdot e^{0.02x}$ (dollars per year, and x is measured in years) at an annual interest rate of 5%, compounded continuously.

Solution: The capital value is given by

$$\int_0^\infty e^{-rx} f(x)\,dx = \int_0^\infty e^{-0.05x}\,12\,000 \cdot e^{0.02x}\,dx = 12\,000 \cdot \frac{e^{-0.03x}}{-0.03}\Big|_0^\infty = 400\,000 \text{ (dollars)}.$$

(b) (5 points) You have the choice of renting a house with the above money flow paid as rent, or buying it for \$ 150 000. Would you buy it if you had this amount available? Explain. (Do *not* take taxes and other costs of house-owners into account.)

Solution: The price offered is less than the capital value of the flow, hence I would definitely buy the house. For a more detailed argument, if I want to deposit an initial amount on my bank account, which will cover my house rent, then this amount would be the capital value \$400 000. Doing so, I don't have to worry about the house rent anymore.

On the other hand, by paying \$150 000, I could buy the house and then, of course, I don't have to worry about the house rent anymore. Comparing the two cases, buying the house only requires \$150 000, while paying the rent from interests of an initial amount requires \$400 000.

The story as seen by the present house owner: if I pay the house rent, then on the long run the house owner will have the same amount on her/his bank account as if I would pay her/him an initial sum of \$400 000. As opposed to this, I only pay \$150 000 if I buy the house so, in the long run, s/he will see less money from me in this case.

3.(a) (5 points) Compute

$$\int_0^\infty x \cdot e^{-2x}\, dx.$$

Solution: Integration by parts with $u = x$, $v = \frac{e^{-2x}}{-2}$ leads to

$$\int_0^\infty x \cdot e^{-2x}\, dx = \left[x \cdot \frac{e^{-2x}}{-2} \right]\Bigg|_0^\infty - \int_0^\infty \frac{e^{-2x}}{-2}\, dx = \frac{1}{2} \int_0^\infty e^{-2x}\, dx = \frac{1}{2} \cdot \frac{e^{-2x}}{-2}\Bigg|_0^\infty = \frac{1}{4}.$$

(b) (5 points) Compute

$$\int_0^\infty \frac{e^{2x}}{1+e^{2x}}\, dx.$$

Solution: Substitute $u = 1 + e^{2x}$ to compute

$$\int_0^\infty \frac{e^{2x}}{1+e^{2x}}\, dx = \frac{1}{2} \int_0^\infty \frac{1}{1+e^{2x}}\, [2e^{2x} dx] = \frac{1}{2} \int_2^\infty \frac{1}{u}\, du = \frac{1}{2} \ln(|u|)\Bigg|_2^\infty = \infty.$$

(c) (5 points) Compute

$$\int_0^\infty \frac{e^{2x}}{(1+e^{2x})^2}\,dx.$$

Solution: Substitute $u = 1 + e^{2x}$ to compute

$$\int_0^\infty \frac{e^{2x}}{(1+e^{2x})^2}\,dx = \frac{1}{2}\int_0^\infty \frac{1}{(1+e^{2x})^2}\,[2e^{2x}dx] = \frac{1}{2}\int_2^\infty \frac{1}{u^2}\,du = \frac{1}{2}\cdot\frac{u^{-1}}{-1}\bigg|_2^\infty = \frac{1}{4}.$$

4. The production function of a certain item is given by $P(x,\,y) = x^{0.75}y^{0.25}$, where x is labor (in hundred work-hours per week) and y is capital (in thousand dollars) invested. Use total differentials to approximate

(a) (10 points) the increase of production when labor is changed from 16 to 17 (hundred work-hours per week) and capital is changed from 81 to 83 (thousand dollars),

Solution:

$$\begin{aligned} dP \simeq P_x(x,\,y)\,dx + P_y(x,\,y)\,dy &= 0.75\cdot\frac{1}{x^{0.25}}\cdot y^{0.25}\,dx + 0.25\cdot x^{0.75}\cdot\frac{1}{y^{0.75}}\,dy \\ &= 0.75\cdot\frac{1}{16^{0.25}}\cdot 81^{0.25}\cdot 1 + 0.25\cdot 16^{0.75}\cdot\frac{1}{81^{0.75}}\cdot 2 \\ &= 0.75\cdot\frac{1}{2}\cdot 3\cdot 1 + 0.25\cdot 8\cdot\frac{1}{27}\cdot 2 = \frac{9}{8} + \frac{4}{27} \simeq 1.27. \end{aligned}$$

(b) (5 points) the relative increase of production when labor is increased by 2% but capital is decreased by 6%.

Solution: The relative increase is

$$\begin{aligned} \frac{dP}{P} \simeq \frac{0.75\cdot\frac{1}{x^{0.25}}\cdot y^{0.25}\,dx + 0.25\cdot x^{0.75}\cdot\frac{1}{y^{0.75}}\,dy}{x^{0.75}y^{0.25}} &= 0.75\cdot\frac{dx}{x} + 0.25\cdot\frac{dy}{y} \\ &= 0.75\cdot 0.02 + 0.25\cdot(-0.06) = 0. \end{aligned}$$

The production will not change under these circumstances. Notice that this computation did not use the actual value of x or y, only the relative change of these quantities.