Math 213 Second In-class Exam

Version B

Please show your computations, not only your answers.

Room B239, 9:55am - 10:45am, March 18, 2005 **NAME:**

Márton Balázs **Dis. Session:**

1.(a) (5 points) Compute

$$\int_0^\infty x\cdot e^{-3x}\,dx.$$

Solution: Integration by parts with $u=x$, $v=\frac{e^{-3x}}{-3}$ leads to

$$\int_0^\infty x\cdot e^{-3x}\,dx=\left[x\cdot\frac{e^{-3x}}{-3}\right]\Bigg|_0^\infty-\int_0^\infty\frac{e^{-3x}}{-3}\,dx=\frac{1}{3}\int_0^\infty e^{-3x}\,dx=\frac{1}{3}\cdot\frac{e^{-3x}}{-3}\bigg|_0^\infty=\frac{1}{9}.$$

(b) (5 points) Compute

$$\int_0^\infty\frac{e^{3x}}{1+e^{3x}}\,dx.$$

Solution: Substitute $u=1+e^{3x}$ to compute

$$\int_0^\infty\frac{e^{3x}}{1+e^{3x}}\,dx=\frac{1}{3}\int_0^\infty\frac{1}{1+e^{3x}}\,[3e^{3x}dx]=\frac{1}{3}\int_3^\infty\frac{1}{u}\,du=\frac{1}{3}\ln(|u|)\bigg|_3^\infty=\infty.$$

(c) (5 points) Compute

$$\int_0^\infty\frac{e^{3x}}{(1+e^{3x})^2}\,dx.$$

Solution: Substitute $u=1+e^{3x}$ to compute

$$\int_0^\infty\frac{e^{3x}}{(1+e^{3x})^2}\,dx=\frac{1}{3}\int_0^\infty\frac{1}{(1+e^{3x})^2}\,[3e^{3x}dx]=\frac{1}{3}\int_2^\infty\frac{1}{u^2}\,du=\frac{1}{3}\cdot\frac{u^{-1}}{-1}\bigg|_2^\infty=\frac{1}{6}.$$

2. (15 points) What is the minimum length of a fence **used on three** out of the four **sides** of a rectangular property having an area of 2000 square yards?

Solution: Let the two parallel edges of the property with fences have each length x, and the remaining edge with a fence have length y. Then we have to minimize $f(x, y) = 2x + y$ subject to the constraint $xy = 2000$, or $g(x, y) = xy - 2000 = 0$. Consider the Lagrange function and its first-order derivatives:

$$\begin{aligned} F(x, y, \lambda) &= 2x + y - \lambda(xy - 2000), \\ F_x(x, y, \lambda) &= 2 - \lambda y, \\ F_y(x, y, \lambda) &= 1 - \lambda x, \\ F_\lambda(x, y, \lambda) &= -xy + 2000. \end{aligned}$$

The length of the fence must have a minimum, and the corresponding (x, y) pair is found among the solutions of the system obtained by making the last three equations zero. From $F_x = 0$, we conclude $\lambda = 2/y$, from $F_y = 0$ we derive $\lambda = 1/x$, therefore we obtain $2/y = 1/x$, or $y = 2x$. Plugging this back to $F_\lambda = 0$, we have

$$\begin{aligned} -x \cdot 2x + 2000 &= 0 \\ 2x^2 &= 2000 \\ x &= \sqrt{1000}. \end{aligned}$$

Hence $y = 2x = 2\sqrt{1000}$, and the minimum length of the fence is

$$f(\sqrt{1000}, 2\sqrt{1000}) = 2\sqrt{1000} + 2\sqrt{1000} \simeq 126.5 \text{ (yards)}.$$

3. The production function of a certain item is given by $P(x, y) = x^{2/3}y^{1/3}$, where x is labor (in hundred work-hours per week) and y is capital (in thousand dollars) invested. Use total differentials to approximate

(a) (10 points) the increase of production when labor is changed from 27 to 28 (hundred work-hours per week) and capital is changed from 64 to 66 (thousand dollars),

Solution:

$$\begin{aligned} dP \simeq P_x(x, y)\,dx + P_y(x, y)\,dy &= \frac{2}{3} \cdot \frac{1}{x^{1/3}} \cdot y^{1/3}\,dx + \frac{1}{3} \cdot x^{2/3} \cdot \frac{1}{y^{2/3}}\,dy \\ &= \frac{2}{3} \cdot \frac{1}{27^{1/3}} \cdot 64^{1/3} \cdot 1 + \frac{1}{3} \cdot 27^{2/3} \cdot \frac{1}{64^{2/3}} \cdot 2 \\ &= \frac{2}{3} \cdot \frac{1}{3} \cdot 4 \cdot 1 + \frac{1}{3} \cdot 9 \cdot \frac{1}{16} \cdot 2 = \frac{8}{9} + \frac{6}{16} \simeq 1.26. \end{aligned}$$

(b) (5 points) the relative increase of production when labor is increased by 2% but capital is decreased by 4%.

Solution: The relative increase is

$$\frac{dP}{P} \simeq \frac{\frac{2}{3}\cdot\frac{1}{x^{1/3}}\cdot y^{1/3}\,dx + \frac{1}{3}\cdot x^{2/3}\cdot\frac{1}{y^{2/3}}\,dy}{x^{2/3}y^{1/3}} = \frac{2}{3}\cdot\frac{dx}{x} + \frac{1}{3}\cdot\frac{dy}{y}$$
$$= \frac{2}{3}\cdot 0.02 + \frac{1}{3}\cdot(-0.04) = 0.$$

The production will not change under these circumstances. Notice that this computation did not use the actual value of x or y, only the relative change of these quantities.

4.(a) (10 points) Find the capital value (i.e. the present value up to an infinite time) of a continuous money flow flowing with rate $f(x) = 8\,000 \cdot e^{0.03x}$ (dollars per year, and x is measured in years) at an annual interest rate of 7%, compounded continuously.

Solution: The capital value is given by

$$\int_0^\infty e^{-rx}\,f(x)\,dx = \int_0^\infty e^{-0.07x}\,8\,000\cdot e^{0.03x}\,dx = 8\,000\cdot\left.\frac{e^{-0.04x}}{-0.04}\right|_0^\infty = 200\,000 \text{ (dollars)}.$$

(b) (5 points) You have the choice of renting a house with the above money flow paid as rent, or buying it for \$ 150 000. Would you buy it if you had this amount available? Explain. (Do *not* take taxes and other costs of house-owners into account.)

Solution: The price offered is less than the capital value of the flow, hence I would definitely buy the house. For a more detailed argument, if I want to deposit an initial amount on my bank account, which will cover my house rent, then this amount would be the capital value \$200 000. Doing so, I don't have to worry about the house rent anymore.

On the other hand, by paying \$150 000, I could buy the house and then, of course, I don't have to worry about the house rent anymore. Comparing the two cases, buying the house only requires \$150 000, while paying the rent from interests of an initial amount requires \$200 000.

The story as seen by the present house owner: if I pay the house rent, then on the long run the house owner will have the same amount on her/his bank account as if I would pay her/him an initial sum of \$200 000. As opposed to this, I only pay \$150 000 if I buy the house so, in the long run, s/he will see less money from me in this case.