Math 320 Sample First In-class Exam

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1. Above a certain velocity, the acceleration of vehicles pulled by electric motors (streetcars, trains, trolleybuses, electric cars) can be approximated to be proportional to the inverse of the square of their speed. Assume hence, by neglecting friction and air resistance, that an elevated train traveling with speed v has acceleration

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{100}{v^2}$$

in the meter-second system.

(a) (20 points) If at time zero it passes the end of the platform with a speed of 10 (m/s) (approximately 22.5 mi/h), then find the **distance** (not the time!) in meters needed to accelerate to 20 (m/s) (approx. 45 mi/h), 30 (m/s) (approx. 67.5 mi/h) and 40 (m/s) (approx. 90 mi/h).

(b) (15 points) Show that its speed tends to infinity as $t \to \infty$. In other words, neglecting friction and air resistance, an electric vehicle accelerates to arbitrary large speeds, just it takes a long way, as seen in question (a).

2. (30 points) Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\mathrm{e}^{-x^2}, \qquad y(0) = 1.$$

3. For

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin y \cdot \cos y$$

(a) (15 points) Sketch, roughly, a direction field and classify the critical points.

(b) (10 points) Determine, from your sketch, the asymptotic behavior of the solution for $y(0) = 1$ as $t \to \infty$.

4. Solve the system

$$\begin{aligned} 3x_1 + x_2 + 2x_3 &= 4 \\ x_2 - x_3 &= 4 \\ -2x_1 + 4x_2 - x_3 &= 11 \end{aligned}$$

using

(a) (20 points) Gauss-Jordan elimination,

(b) (20 points) Cramer's method.

5. Given the matrix

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

(a) (15 points) compute $\mathbf{A}^2$ and $\mathbf{A}^3$.

(b) (15 points) Based on these results, determine the matrices $\mathbf{A}^{-1}$ and $\mathbf{A}^{2004}$.