Math 320. LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS
First midterm exam. October 14, 2002.

1. Solve the following differential equation:

$$xy' + y = e^{2x}.$$

Find the solution of the preceding equation that satisfies the initial condition $y(1) = 0$.

(2 points)

Solution We write first the equation in the standard form: $y' + y/x = e^{2x}/x$. Now we put $y = uv$ so that the equation reads $u'v + u(v' + v/x) = e^{2x}/x$. Then we choose v so that $v' + v/x = 0$. By separating variables, this is $dv/v = -dx/x$, so integration yields, $\log v = -\log x = \log(1/x)$ and $v = 1/x$. Once v has been chosen, we solve the equation in u, which is now $u'/x = e^{2x}/x$, that is, $u' = e^{2x}$. By direct integration, $u = e^{2x}/2 + C$. The general solution of the equation is then

$$y = uv = \frac{e^{2x}}{2x} + \frac{C}{x}.$$

If we want $y(1)$ to be 0, we have to pick C so that $0 = e^2/2 + C$, that is $C = -e^2/2$ and the required solution is

$$y = \frac{e^{2x} - e^2}{2x}.$$

Common mistakes: (1) From $\log v = -\log x$ get the wrong result $v = -x$. (2) Forget about the constant C.

2. As a consequence of an epidemic in a population of rabbits, the birth rate is 0 and the death rate is proportional to the population itself.

(i) Write the differential equation for the function P that gives the number of rabbits in the population at the time t.

(ii) Solve the differential equation in (i) and give an explicit expression for $P(t)$. (Assume that the initial population is P_0.) According to this solution, which is the behaviour of the population in the long run?

(iii) Suppose we count the time in months. If the initial population is 2000 rabbits and 5 months later is only 1000, how many rabbits will survive after 15 months? Which is the rate of deaths per month at this moment?

(3 points)

Solution (i) The general equation is $P' = (\beta - \delta)P$, where β and δ are the birth and death rates, respectively. In our problem, $\beta = 0$ and $\delta = kP$ for some constant k (this is because death rate is proportional to P). So the equation is $P' = -kP^2$.

(ii) We solve the equation by separating variables: $dP/P^2 = -kdt$. By direct integration, $-1/P = -kt + C$ or, changing the sign, $1/P = kt + C$ (C is a different constant, but it doesn't matter). Since the initial population is P_0, we must have $1/P_0 = C$, so $1/P = kt + (1/P_0)$. Multiplying by P_0 and taking reciprocals we obtain the desired function:

$$P(t) = \frac{P_0}{1 + P_0 kt}.$$

As t goes to $+\infty$, the denominator (which never vanishes) also goes to $+\infty$, so $P(t)$ goes to zero. This means that the population will eventually become extinct.

(iii) The condition $P(5) = 1000$ tells us which is the value of k: $1000 = 2000/(1 + 10000k)$, so $k = 0.0001$. Then the population after 15 months is $P(15) = 2000/(1 + 2000 \cdot 0.0001 \cdot 15) = 500$. The rate of deaths is the rate at which the population is decreasing, so we are being asked for the value $P'(15)$, which is $P'(15) = -k \cdot 500^2 = -0.0001 \cdot 500^2 = -25$.

Common mistakes: (1) Write the equation $P' = -kP$. (2) Forget about the constant C when solving the equation or working it out incorrectly.

3. Suppose a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides a deaceleration of 0.1 times the velocity of the car.

(i) Write the differential equation for the velocity and find explicitly the velocity of the car as a function of the time.
(ii) Find the car's limiting velocity.
(iii) Find how long it takes to the car to attain 90% of its limiting velocity.

(2.5 points)

Solution (i) The net acceleration is $10 - 0.1v$, so the differential equation is $v' = 10 - 0.1v$. We can solve it by separating the variables (also as a first order linear equation). Thus, $dv/(10-0.1v) = dt$ and, by integration, $-10\log(10-0.1v) = t + C$, so $10 - 0.1v = Ce^{-t/10}$ and $v(t) = 100 + Ce^{-t/10}$. Since the car starts from rest, $v(0) = 0$, so $C = -100$ and the solution is

$$v(t) = 100(1 - e^{-t/10}).$$

(ii) When t goes to infinity, the negative exponential $e^{-t/10}$ goes to zero, so the limit velocity is 100.

(iii) The 90% of the limiting velocity is 90, so we need to solve the equation $90 = 100(1 - e^{-t/10})$. The solution is $t = 10\log 10$.

4. Let a be a parameter and consider the matrix depending on this parameter:

$$\begin{pmatrix} a & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

(i) Evaluate the determinant of this matrix. Find the value of a for which this matrix is **not** invertible.
(ii) Find the inverse of the matrix when $a = 1$. (Use the method you prefer.)
(iii) Using (ii) find the solution of the linear system:

$$\begin{cases} x + y - z = 1 \\ 2x + y = 1 \\ x - y + 2z = 1. \end{cases}$$

(2.5 points)

Solution

(i) The determinant of the matrix is $2a - 1$. So the matrix is not invertible when $2a - 1 = 0$, that is $a = 1/2$.

(ii) Use elementary row transformations or calculate the adjoint matrix (then transpose it and finally divide by the determinant). The result is

$$\begin{pmatrix} 2 & -1 & 1 \\ -4 & 3 & -2 \\ -3 & 2 & -1 \end{pmatrix}.$$

(iii) The solution of this system is

$$A^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix},$$

that is $x = 2$, $y = -3$, $z = -2$.