**Math 320. LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS**
Second midterm exam. November 25, 2002.

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**1.** Consider the three vectors in $\mathbb{R}^4$,

$$u_1 = (1, -1, -3, 2), \quad u_2 = (-1, 1, 2, -1), \quad u_3 = (-2, 2, 1, 1).$$

(i) Are they linearly independent? If not, find a non-trivial linear combination of them equal to zero. Which is a basis for the subspace spanned by the three vectors $u_1$, $u_2$, $u_3$?

(ii) Let $v_1 = u_1 + u_3$ and $v_2 = u_2 + 2u_3$. Are they linearly independent? Write, if possible, $u_3$ as a linear combination of $v_1$ and $v_2$. What is the relation between the subspaces spanned by $u_1, u_2, u_3$ and by $v_1, v_2$?

(1.5 points)

**Solution** (i) To see if the vectors $u_1$, $u_2$, $u_3$ are linearly independent we study the relation $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$. If the only possibility for the $\lambda$'s is $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then the vectors are linearly independent. Otherwise, they are linearly dependent and we can find explicitly a non-trivial relation among them. We have

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = (\lambda_1 - \lambda_2 - 2\lambda_3, -\lambda_1 + \lambda_2 + 2\lambda_3, -3\lambda_1 + 2\lambda_2 + \lambda_3, 2\lambda_1 - \lambda_2 + \lambda_3),$$

so we have to solve the homogeneous linear system:

$$\begin{cases} \lambda_1 & - & \lambda_2 & - & 2\lambda_3 & = & 0 \\ -\lambda_1 & + & \lambda_2 & + & 2\lambda_3 & = & 0 \\ -3\lambda_1 & + & 2\lambda_2 & + & \lambda_3 & = & 0 \\ 2\lambda_1 & - & \lambda_2 & + & \lambda_3 & = & 0. \end{cases}$$

To solve this system you can either write the coefficient matrix and apply elementary row transformations or simply note that adding the second and last equation one gets $\lambda_1 = -3\lambda_3$. Then, from the first equation, $\lambda_2 = -5\lambda_3$. One can check now that for any $t$, $\lambda_1 = -3t$, $\lambda_2 = -5t$ and $\lambda_3 = t$ is a solution. In particular, there are non-trivial solutions (for instance $\lambda_1 = 3$, $\lambda_2 = 5$, $\lambda_3 = -1$), thus the three vectors are linearly dependent and a non-trivial relation among them is

$$3u_1 + 5u_2 - u_3 = 0.$$

Since $u_3$ is a linear combination of $u_1$ and $u_2$ (we have just seen that $u_3 = 3u_1 + 5u_2$) it follows that the subspace spanned by $u_1$, $u_2$ and $u_3$ is the same as for only $u_1$ and $u_2$. On the other hand these two vectors are linearly independent (they are not proportional), so $u_1$, $u_2$ is a basis for the subspace spanned by the three vectors. (Other possible solutions are $u_2$, $u_3$ and $u_1$, $u_3$.)

(ii) We have that $v_1 = (-1, 1, -2, 3)$ and $v_2 = (-5, 5, 4, 1)$. They are not proportional, so they are linearly independent (or, if you like, check that the only way of having $\lambda_1 v_1 + \lambda_2 v_2 = 0$ is when $\lambda_1 = \lambda_2 = 0$).

To see if $u_3$ is a linear combination of $v_1$ and $v_2$, we try to find $\lambda_1$ and $\lambda_2$ such that $\lambda_1 v_1 + \lambda_2 v_2 = u_3$. This yields the linear system:

$$\begin{cases} -\lambda_1 & - & 5\lambda_2 & = & -2 \\ \lambda_1 & + & 5\lambda_2 & = & 2 \\ -2\lambda_1 & + & 4\lambda_2 & = & 1 \\ 3\lambda_1 & + & \lambda_2 & = & 1. \end{cases}$$

This system is consistent and its (unique) solution is $\lambda_1 = 3/14$, $\lambda_2 = 5/14$, so $u_3 = (3/14)v_1 + (5/14)v_2$.

The subspace spanned by $u_1$, $u_2$, $u_3$ has dimension 2 (we have found a basis in part (i) with two vectors) and $v_1$, $v_2$ are contained in this subspace (because they are linear combinations of the $u_i$'s) and are linearly independent, so $v_1$, $v_2$ is also a basis of the subspace spanned by the $u_i$'s. This means that both the subspace spanned by the $u_i$'s and the subspace spanned by the $v_j$'s are actually the same.

**2.** (i) Find the general solution of the linear differential equation:

$$y''' - 2y' + 4y = 8e^{-2x}.$$

(ii) Explain (do not make any computation) how you would solve the equations $y''' - 2y' + 4y = x \sin x$ and $y''' - 2y' + 4y = (1 + x)e^{-2x}$.

(3 points)

**Solution** (i) The characteristic equation is $x^3 - 2x + 4 = 0$. One solution of this equation is $-2$ (by inspection). Dividing by $x + 2$, we find that $x^3 - 2x + 4 = (x + 2)(x^2 - 2x + 2)$, so the other solutions of the characteristic equation are $1 \pm i$. We conclude that the general solution of the associated homogeneous equation is

$$y_c(x) = c_1 e^{-2x} + c_2 e^x \cos x + c_3 e^x \sin x.$$

To find a particular solution of the equation we try with a function of the form $y_p(x) = Axe^{-2x}$ (we don't try with something like $Ae^{-2x}$ because this is already a solution of the homogeneous equation). After computing the first three derivatives of $y_p$ we find that $y_p'''(x) - 2y_p'(x) + 4y_p(x) = 10Ae^{-2x}$, so we simply need to take $A = 4/5$. So, the general solution of the equation is:

$$y(x) = c_1 e^{-2x} + c_2 e^x \cos x + c_3 e^x \sin x + \frac{4}{5} x e^{-2x}.$$

(ii) We argue similarly as in the last part of (i). For the case $x \sin x$ we try with a function of the form $y_p(x) = A \sin x + Bx \sin x + C \cos x + Dx \cos x$. In the second case we try with $y_p(x) = Axe^{-2x} + Bx^2 e^{-2x}$ (simply $Ae^{-2x} + Bxe^{-2x}$ doesn't work because of duplication).

**3.** A body with mass $m = 1$ is attatched to a spring with constant $k = 4$ (in a suitable unit system). The spring is initially stretched to the right 1 m. and then is pushed to the right so that the body starts moving with velocity 2 m/s.

(i) Find the function of the motion and write it in the form $x(t) = C \cos(\omega t + \alpha)$ (neglect friction). Which is the amplitude, frequency and phase angle of the motion? Draw a sketch of the graph of the function $x(t)$.

(ii) Consider the same situation as in (i) but now suppose there exists friction, which is proportional to the velocity and has constant $c = 5$. Find the function $x(t)$ of the motion in this case. Which is the maximum distance to the right from the equilibrium position reached by the body? How long does it take to reach this point? Draw a sketch of the function $x(t)$ in this case.

(3 points)

**Solution** (i) The equation of the motion is $x'' + 4x = 0$ with general solution $x(t) = c_1 \cos 2t + c_2 \sin 2t$. The initial conditions $x(0) = 1$ and $x'(0) = 2$ imply that $c_1 = c_2 = 1$. Drawing the rectangle triangle with sides 1, 1 and hypotenuse $\sqrt{2}$ we see that $1 = \sqrt{2} \cos \alpha$ and also $1 = \sqrt{2} \sin \alpha$, where $\alpha = \pi/4$. Then

$$x(t) = \sqrt{2}(\cos \pi/4 \cos 2t + \sin \pi/4 \sin 2t) = \sqrt{2} \cos(2t - \pi/4).$$

Thus the amplitude of the motion is $\sqrt{2}$, the frequency is $2/2\pi = 1/\pi$ and the phase angle is $\alpha = \pi/4$.

(ii) In this case the equation of the motion is $x'' + 5x' + 4x = 0$. The solutions for the characteristic equation are $-1$ and $-4$, so the general solution of the equation is $x(t) = c_1 e^{-t} + c_2 e^{-4t}$. The initial conditions $x(0) = 1$, $x'(0) = 2$ imply that $c_1 = 2$ and $c_2 = -1$, so the function of the motion is $x(t) = 2e^{-t} - e^{-4t}$.

To find the maximum distance to the right travelled by the body we solve the equation $x'(t) = 0$, that is, $-2e^{-t} + 4e^{-4t} = 0$ or $e^{-t} = 2e^{-4t}$. Taking logarithms we obtain $-t = \log 2 - 4t$, so the maximum is attained when $t = (\log 2)/3 = \log 2^{1/3}$ and the maximum distance is

$$x(\log 2^{1/3}) = 2 \cdot 2^{-1/3} - 2^{-4/3} = \frac{3}{2\sqrt[3]{2}}.$$

**4.** Let $A$ be the matrix:

$$A = \begin{pmatrix} -5 & -6 & 2 \\ 4 & 5 & -2 \\ 2 & 3 & -2 \end{pmatrix}$$

(i) Find, if possible, an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.
(ii) Find the matrix $A^{51}$. (You can answer this question without performing explicitly any matrix multiplication or matrix inversion at all.)

(2.5 points)

**Solution** (i) The characteristic polynomial of the matrix $A$ is $x(x+1)^2$, so there are only two eigenvalues, namely, 0 and $-1$ (this one is double). To find the eigenvectors corresponding to 0 we solve the homogeneous system $AX = 0$. The solutions are $x = 2t$, $y = -2t$, $z = -t$, so a basis for this eigenspace is $(2, -2, -1)$. For the eigenvalue $-1$ we have to solve $(A + I)X = 0$, which reduces to the equation $2x + 3y - z = 0$. The solutions are $x = r$, $y = s$, $z = 2r + 3s$, so a typical solution is $(r, s, 2r + 3s) = r(1, 0, 2) + s(0, 1, 3)$ and a basis for this eigenspace is $(1, 0, 2)$, $(0, 1, 3)$. Notice that the dimension is 2, which coincides with the multiplicity of the eigenvalue $-1$, so the matrix is diagonalizable. Actually, if

$$P = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then $P^{-1}AP = D$.

(ii) By (i), $A = PDP^{-1}$, so $A^{51} = PD^{51}P^{-1}$. But it is clear that $D^{51} = D$, so $A^{51} = PDP^{-1} = A$.