Math 320. LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

Final exam. December 15, 2002.

1. Let A be the matrix:

$$A = \begin{pmatrix} -2 & 1 & 0 & 2 \\ -3 & 2 & 0 & 2 \\ 1 & -1 & -1 & -2 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

(i) Prove that the matrix A is invertible. If B is any 4×1 column vector, how many solutions does the system $AX = B$ have?

(ii) The characteristic polynomial of A is $(x^2 - 1)^2$ (you don't need to prove this). For each eigenvalue, find a basis for the corresponding eigenspace. Is the matrix A diagonalizable?

(2 points)

Solution

(i) The easiest way to show that A is invertible is checking that its determinant is different from 0. To compute $\det A$ we expand the determinant along the third column. We get

$$\det A = - \begin{vmatrix} -2 & 1 & 2 \\ -3 & 2 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 1,$$

thus A is invertible. Then to solve the linear system of equations $AX = B$ we can multiply everything by A^{-1} and we get that $X = A^{-1}B$ is the unique solution of the system so any system $AX = B$ has exactly one solution.

(ii) Since the characteristic polynomial of A is $(x^2 - 1)^2$, we have that the eigenvalues of A are 1 and -1, both of them with multiplicity 2. To compute the eigenvectors of the eigenvalue -1 we solve the system $(A + I)X = 0$. The first, third and fourth equations are basically the same: $x - y - 2t = 0$. Taking into account also the second equation $-3x + 3y + 2t = 0$ we get that $t = 0$ and $x = y$. Thus any solution of this system has the form $(x, x, z, 0) = x(1, 1, 0, 0) + z(0, 0, 1, 0)$. This means that the eigenspace corresponding to the eigenvalue -1 is the subspace spanned by the vectors $(1, 1, 0, 0)$ and $(0, 0, 1, 0)$, which are linearly independent, so one basis for this eigenspace is $(1, 1, 0, 0)$, $(0, 0, 1, 0)$.

Similarly, for the eigenvalue 1 we have to solve $(A - I)X = 0$. One can check that the solutions of this system have the form $(x, x, -x, x) = x(1, 1, -1, 1)$, so a basis for this eigenspace is $(1, 1, -1, 1)$. In particular its dimension is 1, which is strictly smaller than the multiplicity of the eigenvalue, which is 2. We conclude that the matrix A is *not* diagonalizable.

2. (i) Find the general solution of the differential equation

$$x'' + 2x' + 5x = 5 \sin t.$$

(ii) Suppose the equation $x'' + 2x' + 5x = 0$ describes the motion of a mass attached to a spring with resistance proportional to the velocity. The spring is initially stretched to the right 1 m. and then pushed to the right so that the initial velocity is 3 m/s. Find the function of the motion $x = x(t)$ in this case and draw a sketch of this function. The maxima and minima of this function lie on a pair of symmetric curves that "envelope" the function x. Which functions are these?

(iii) Suppose now that in addition to the forces due to the spring and the friction, there is an external force $F(t) = 5 \sin t$, so that the equation of the motion is the one in (i). How will the mass move in the long run? (You don't need to do any computation here, just interpret the solution in (i)).

(3 points)

Solution

(i) The characteristic equation of this differential equation is $x^2+2x+5=0$ and the solutions of this equation are $-1 \pm 2i$. This means that the general solution of the homogeneous equation $x'' + 2x' + 5x = 0$ is

$$x_c(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t = e^{-t}(c_1 \cos 2t + c_2 \sin 2t).$$

Now, to find a particular solution of the non-homogeneous equation we try with a function of the form $x_p(t) = A\cos t + B \sin t$. After computing the derivatives $x_p'(t)$ and $x_p''(t)$ we get that

$$x_p''(t) + 2x_p'(t) + 5x(t) = (4A + 2B)\cos t + (-2A + 4B)\sin t.$$

Since we want x_p to be a solution of the equation we need $4A + 2B = 0$ and $-2A + 4B = 5$. This yields $A = -1/2$ and $B = 1$. The general solution of the equation is then

$$x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) - \frac{1}{2}\cos t + \sin t.$$

(ii) The general solution of the homogeneous equation $x'' + 2x' + 5x = 0$ has been calculated in (i): $x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$, so $1 = x(0) = c_1$. We compute $x'(t)$ and evaluate it for $t = 0$. We get $x'(0) = -c_1 + 2c_2$. Since $x'(0) = 3$ and $c_1 = 1$, we obtain that $c_2 = 2$, thus the equation for the motion of the body is

$$x(t) = e^{-t}(\cos 2t + 2\sin 2t).$$

Consider the rectangle triangle with sides 1 and 2 and hypotenuse $\sqrt{5}$. Then if α is the angle between the hypotenuse and the side of lenght 1 we have that $1 = \sqrt{5}\cos\alpha$ and $2 = \sqrt{5}\sin\alpha$, thus

$$\cos 2t + 2\sin 2t = \sqrt{5}(\cos\alpha \cos 2t + \sin\alpha \sin 2t) = \sqrt{5}\cos(2t - \alpha)$$

and the function $x(t)$ can be written as

$$x(t) = \sqrt{5}e^{-t}\cos(2t - \alpha).$$

This means that the function $x(t)$ oscillates between the two symmetric functions $f(t) = \pm\sqrt{5}e^{-t}$. Notice that the amplitude of the oscillations decrease to zero, so the limit of $x(t)$ as t goes to infinity is also 0.

(iii) As we have seen in (i), the function describing the motion is

$$x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) - \frac{1}{2}\cos t + \sin t.$$

As t increases the negative exponential e^{-t} goes to zero very fast, whereas the expression $c_1 \cos 2t + c_2 \sin 2t$ remains bounded. This means that the product $e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ becomes very small and can be neglected. Thus for t large, the motion can be accurately described simply by the function $x_1(t) = -\frac{1}{2}\cos t + \sin t$, which corresponds to a simple harmonic motion with frequency $\frac{1}{2\pi}$ and amplitude $\sqrt{5}/2$.

3. (i) Give the general solution of the linear system of differential equations:

$$\begin{cases} x' = -3x + 2y \\ y' = -2x + 2y. \end{cases}$$

(ii) Draw a sketch of the trajectories of the solutions of the system in (i). Is the equilibrium solution $x(t) = 0$, $y(t) = 0$ stable or unstable? Explain what this means.

(iii) Find now the general solution of the non-homogeneous system:

$$\begin{cases} x' &= -3x &+ 2y &+ 3e^{-t} \\ y' &= -2x &+ 2y. \end{cases}$$

(3 points)

Solution

(i) This system can be written as $X' = AX$ with $A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$. The eigenvalues of A are 1 and -2 and the corresponding eigenvectors are $(1,2)$ and $(2,1)$. Thus if $P = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ we have $P^{-1}AP = D$ or $A = PDP^{-1}$. Thus the system we have to solve is $X' = AX = PDP^{-1}X$, or $P^{-1}X' = DP^{-1}X$. At this point we make the substitution $Y = P^{-1}X$ so that now the system reads $Y' = DY$. This system is $y_1' = y_1$ and $y_2' = -2y_2$ whose solutions are $y_1(t) = c_1e^t$ and $y_2(t) = c_2e^{-2t}$, so the general solution of the system is

$$X(t) = PY(t) = P\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1e^t \\ c_2e^{-2t} \end{pmatrix} = \begin{pmatrix} c_1e^t + 2c_2e^{-2t} \\ 2c_1e^t + c_2e^{-2t} \end{pmatrix}.$$

(ii) Since one of the eigenvalues is positive and the other is negative the critical point $(0,0)$ is a saddle point. If you draw the lines passing through the origin with the directions indicated by the eigenvectors $(1,2)$ and $(2,1)$, a typical trajectory has these two lines as asymptotes (like in a hyperbola). The equilibrium solution is unstable. This means that when the initial condition is exactly $x(0) = 0$, $y(0) = 0$ the solution is constant: $x(t) = 0$, $y(t) = 0$. However, as soon as there is some small perturbation in these initial conditions (no matter how small), then the behavior of the solution changes completely and actually both values $x(t)$ and $y(t)$ grow arbitrarily large.

(iii) We simply need a particular solution of this system (although virtually at the same cost we can obtain the general solution directly). We write the system as $X' = AX + F$ with $F(t) = \begin{pmatrix} 3e^{-t} \\ 0 \end{pmatrix}$. Again $A = PDP^{-1}$, so $P^{-1}X' = DP^{-1}X + P^{-1}F$. We set $Y = P^{-1}X$ so that the system simplifies to $Y' = DY + P^{-1}F$. At this point we have to compute P^{-1} and multiply it by F. It turns out that $P^{-1} = -(1/3)\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, so $P^{-1}F(t) = \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix}$. Then our new the system reads $y_1' = y_1 - e^{-t}$ and $y_2' = -2y_2 + 2e^{-t}$. Each of these equations can be solved separately. Notice that we can find particular solutions for both equations trying with functions of the form Ae^{-t} (there is no duplication). We find that two particular solutions are $y_1(t) = (1/2)e^{-t}$ and $y_2(t) = 2e^{-t}$ (the general solutions are $y_1(t) = c_1e^t + (1/2)e^{-t}$ and $y_2(t) = c_2e^{-2t} + 2e^{-t}$). Thus a particular solution for the system $X' = AX + F$ is

$$X_p(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} (1/2)e^{-t} \\ 2e^{-t} \end{pmatrix} = \begin{pmatrix} (9/2)e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

Finally, the general solution of the system is

$$X(t) = \begin{pmatrix} c_1e^t + 2c_2e^{-2t} \\ 2c_1e^t + c_2e^{-2t} \end{pmatrix} + \begin{pmatrix} (9/2)e^{-t} \\ 3e^{-t} \end{pmatrix} = \begin{pmatrix} c_1e^t + 2c_2e^{-2t} + (9/2)e^{-t} \\ 2c_1e^t + c_2e^{-2t} + 3e^{-t} \end{pmatrix}.$$

4. (i) Explain how to compute the exponential e^T of a triangular matrix $T = \lambda I_n + T_0$ all of whose elements in the diagonal are equal to λ.

(ii) Find e^{At} for $A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$. (You can use the fact that $A = PTP^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $T = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. The inverse of P is $P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$.)

(iii) Find the solution of the system

$$\begin{cases} x' &= & & - & y \\ y' &= & x & - & 2y \end{cases}$$

which satisfies the initial conditions $x(0) = 2$, $y(0) = -1$.

(2 points)

Solution

(i) Since the matrices λI_n and T_0 commute, we have that $e^T = e^{\lambda I_n} e^{T_0}$. The exponential of λI_n can be computed immediately because it is diagonal (actually, $e^{\lambda I_n} = e^{\lambda} I_n$). On the other hand, to compute e^{T_0} we use the definition of the exponential of a matrix. We have to notice that, given the special form of the matrix T_0 (it is triangular and all the elements in the diagonal are zero), the powers of T_0 are eventually 0, so if $T_0^k = 0$, e^{T_0} can be computed by adding only a finite number of matrices:

$$e^{T_0} = I_n + \frac{1}{1!}T_0 + \frac{1}{2!}T_0^2 + \cdots + \frac{1}{k-1!}T_0^{k-1}.$$

(ii) We have

$$e^{At} = e^{PTP^{-1}t} = e^{PTtP^{-1}} = Pe^{Tt}P^{-1}.$$

Notice that $Tt = \begin{pmatrix} -t & -t \\ 0 & -t \end{pmatrix} = \begin{pmatrix} -t & 0 \\ 0 & -t \end{pmatrix} + \begin{pmatrix} 0 & -t \\ 0 & 0 \end{pmatrix}$. The square of $\begin{pmatrix} 0 & -t \\ 0 & 0 \end{pmatrix}$ is zero, so its exponential is $I_2 + \begin{pmatrix} 0 & -t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$. Thus

$$e^{Tt} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

and

$$e^{At} = Pe^{Tt}P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{pmatrix}.$$

(iii) The coefficient matrix of this system is A, so the unique solution of the system that satisfies the given initial conditions is

$$X(t) = e^{At} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} (3t+2)e^{-t} \\ (3t-1)e^{-t} \end{pmatrix}.$$