Math 431 First Evening Exam

Room B239, 6:00pm - 7:00pm, February 18, 2005
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1. My friend Bob takes the Madison - Nowheretown flight with Windy Airlines today, and I'll fly the same flight tomorrow. This airline has three types, A, B and C, of planes, with 3, 4, and 6 seats per row, respectively. We know that the same plane flies to Nowheretown every day, but we don't know of which type this plane is, so we estimate it could be each of the three types A, B or C with equal chance. We also know that seats are assigned completely randomly to passengers.

(a) (15 points) What is the probability that I will have a window seat tomorrow?

Answer: $\frac{2}{3}\cdot\frac{1}{3}+\frac{2}{4}\cdot\frac{1}{3}+\frac{2}{6}\cdot\frac{1}{3}=\frac{1}{2}$.

(b) (15 points) Later Bob tells me that he had a window seat. Then what is the probability that plane type A, B, or C, respectively, is the one to fly to Nowheretown?

Answer: $\frac{\frac{2}{3}\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{4}{9}$, $\frac{\frac{2}{4}\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{1}{3}$ and $\frac{\frac{2}{6}\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{2}{9}$, respectively.

(c) (15 points) Having known that Bob had a window seat, what is the probability that I will also have a window seat tomorrow?

Answer: $\frac{\left(\frac{2}{3}\right)^2\cdot\frac{1}{3}+\left(\frac{2}{4}\right)^2\cdot\frac{1}{3}+\left(\frac{2}{6}\right)^2\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{29}{54}$.

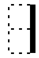
(d) (15 points) Consider the following two events: {I will have a window seat} and {Either type A or type C is flying to Nowheretown}. Are these two events independent? Why?

Answer: Part (b) shows that {I will have a window seat} and {Type B is flying to Nowheretown} are independent events. As {Either type A or type C is flying to Nowheretown} ={Type B is flying to Nowheretown}^{c}, the answer is yes.

2. One out of the nine digits

is randomly chosen and shown on a digital display. Define the following events:

- A is the event that the upper left and the lower right vertical LEDs (and maybe some others as well) are lighting: 

- B is the event that the upper right and the lower right vertical LEDs (and maybe some others as well) are lighting: 
- C is the event that the digit shown on the display is divisible by three.

(a) (15 points) Are A and B independent? Are A and C independent? Are B and C independent? Why?

Answer: $\mathbb{P}\{A\} = \frac{6}{9}$, $\mathbb{P}\{B\} = \frac{6}{9}$, $\mathbb{P}\{C\} = \frac{1}{3}$, $\mathbb{P}\{AB\} = \frac{4}{9}$, $\mathbb{P}\{AC\} = \frac{2}{9}$, $\mathbb{P}\{BC\} = \frac{2}{9}$, thus multiplications show that these three events are pairwise independent.

(b) (15 points) Are $A \cap B$ and C independent? Why?

Answer: $\mathbb{P}\{AB\} = \frac{4}{9}$, $\mathbb{P}\{C\} = \frac{1}{3}$, $\mathbb{P}\{ABC\} = \frac{1}{9}$, multiplication shows that AB and C are not independent.

(c) (15 points) Are A, B and C independent? Why?

Answer: They cannot be independent, as AB and C are not independent by (b).

(d) (15 points) Are A and B conditionally independent, given C? Why?

Answer: $\mathbb{P}\{A \,|\, C\} = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{2}{3}$, $\mathbb{P}\{B \,|\, C\} = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{2}{3}$, while $\mathbb{P}\{AB \,|\, C\} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$, hence the answer is no.

3. (40 points) A centipede has 50 left legs and 50 right legs, and he also has 50 left shoes and 50 right shoes. If he tries on his shoes completely randomly, without respecting left or right, what is the probability that he will have left shoe on each left leg and right shoe on each right leg at the end?

Answer: $\frac{50!\cdot 50!}{100!} = \frac{1}{\binom{100}{50}}$, i.e. compute the number of good permutations over the total number of permutations, or consider the one good selection of all left shoes for the left legs over the $\binom{100}{50}$ possible selections for the 50 left legs.