

**Math 431 Final Exam**

222 Ingraham Hall, 7:45am - 9:45am, May 10, 2005

Márton Balázs

**NAME:**

1. In a country there are four cities  $A$ ,  $B$ ,  $C$  and  $D$ , with four roads  $A \leftrightarrow B$ ,  $B \leftrightarrow C$ ,  $C \leftrightarrow D$  and  $D \leftrightarrow A$ . One night snow blocks each of these roads independently with probability  $p$ .

(a) (20 points) What is the probability that, next morning, city  $A$  will be accessible from city  $B$ ?

(b) (10 points) What is the probability that, next morning, every city will be accessible from every other other city?

2. Let  $F$  be given by

$$F(x) = \begin{cases} 0, & \text{if } x < -1, \\ 1/3, & \text{if } -1 \leq x < 0, \\ 2/3, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } 1 \leq x. \end{cases}$$

(a) (20 points) Show that  $F$  is a probability distribution function.

(b) (20 points) Compute  $\mathbb{E}(X)$  and  $\mathbf{Var}(X)$  for a random variable  $X$  having distribution function  $F$ .

**3.** Let  $X$  be the number shown on a fair die after rolling it. Let  $Y$  be a uniform random variable on the interval  $(0, 1)$ , and independent of  $X$ . Define  $Z := X + Y$ .

**(a)** (20 points) Determine the distribution of  $Z$ .

**(b)** (20 points) Find the expectation and variance of  $Z$ .

4. A fair die is rolled 100 times. Let  $X$  be the number of times when a one is shown on the die. Let  $Y$  be the number of times when a two is shown on the die.

(a) (5 points) Are  $X$  and  $Y$  independent? Why?

(b) (5 points) Are  $X$  and  $Y$  identically distributed?

(c) (10 points) What is the distribution of  $X$ ? What is the distribution of  $Y$ ?

(d) (10 points) What is the distribution of  $X + Y$ ? (Think about what  $X + Y$  counts.)

(e) (10 points) Find  $\mathbb{E}(X)$ ,  $\mathbb{E}(Y)$ ,  $\mathbb{E}(X + Y)$ .

(f) (10 points) Find  $\mathbf{Var}(X)$ ,  $\mathbf{Var}(Y)$ ,  $\mathbf{Var}(X + Y)$ .

- (g) (10 points) Define  $X_i$  as the indicator variable of a one showing up at roll  $i$ ,  $i = 1, \dots, 100$ . Define  $Y_j$  as the indicator variable of a two showing up at roll  $j$ ,  $j = 1, \dots, 100$ . Compute  $\text{Cov}(X_i, Y_j)$  for each pair of indices  $i, j$ .
- (h) (10 points) Use the  $X_i$ 's and  $Y_j$ 's, the properties of the covariance and part (g) to compute  $\text{Cov}(X, Y)$  and  $\text{Corr}(X, Y)$ .
- (i) (10 points) Now compute  $\text{Cov}(X, Y)$  by comparing  $\text{Var}(X) + \text{Var}(Y)$  and  $\text{Var}(X + Y)$  from part (f).
- (j) (10 points) What is the conditional distribution of  $X$ , given  $Y = y$ ? Compute  $\mathbb{E}(X | Y = y)$ .

- (k) (**Bonus problem**, only try when all other problems are completed and checked, 10 points)  
Use the tower rule and the result of part (j) to compute  $\mathbf{Cov}(X, Y)$ .

5. (40 points) A drunk man walks independent exponential ( $\lambda = 1/5$  yards<sup>-1</sup>) distances between consecutive times he falls. (In other words: places he falls on the road form a Poisson process with parameter 1/5.) He gives up after his fiftieth fall. Use the Central Limit Theorem to approximate the probability that he has moved at least 275 yards by that time.