Math 475 second evening exam

3:30 - 4:30 pm, December 3, 2004
Márton Balázs **NAME:**

1. (a) (15 points) Solve the recurrence relation

$$h_n = \frac{5}{2}h_{n-1} - h_{n-2} + \frac{1}{2}n - \frac{1}{2}$$

with initial conditions $h_0 = 2,\ h_1 = 0$.

(b) (5 points) What is the large n-behavior of the solution? And what would be the behavior if we would increase h_1 a little bit?

2. 50 students wrote an exam, and the professor shows the exams to them in his office hour in a large classroom. Each student enters and leaves the classroom precisely once, but can do so any time during that office hour. We assume that the door is small, so no two students can enter or leave simultaneously. Any time the number of students changes in the classroom, we record this number as $0 = h_0,\ h_1,\ h_2,\ \ldots,\ h_{99},\ h_{100} = 0$. Explain how to use the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

to compute

(a) (5 points) the number of such sequences,

(b) (10 points) the number of these sequences such that the professor is never alone in the classroom after the first student entered until the last students leaves,

(c) (5 points) the number of these sequences such that there is at least one time, after the first student entered and before the last students leaves, when the professor is alone in the classroom.

(d) (**Bonus question**, only try when all other problems are completed and the solutions are verified, 6 points): What asymptotic fraction is the answer for (b) and (c), respectively, of the answer for (a) if we have a huge number of students?

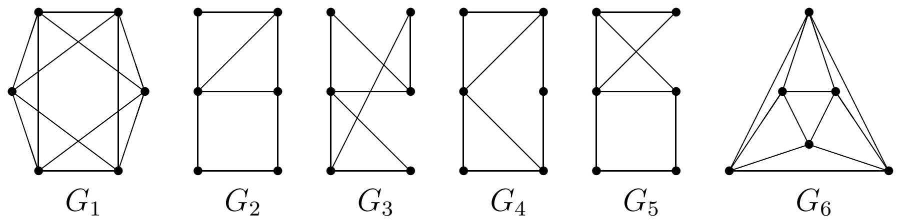
3. (15 points) We know the following data on two number sequences h_n and g_n:

$$h_0 = 3, \quad \Delta h_0 = 4, \quad \Delta^2 h_0 = 1, \quad \Delta^3 h_0 = 2, \quad \Delta^4 h_0 = 7, \quad \Delta^k h_0 = 0,$$
$$g_0 = 6, \quad \Delta g_0 = 7, \quad \Delta^2 g_0 = 2, \quad \Delta^3 g_0 = 4, \quad \Delta^4 g_0 = 14, \quad \Delta^k g_0 = 0$$

for any $k \geq 5$. Compute the precise value of $2h_n - g_n$. (Hint: Think instead of spending too much time on computations.)

4. (a) (5 points) Show that "being isomorphic" is an equivalence relation on the set of graphs.

(b) (10 points) Let S be the set containing the following six connected graphs G_1 to G_6:



Considering two graphs equivalent iff they are isomorphic, find the equivalence classes of S and show the isomorphisms between elements belonging to the same classes.

(c) (5 points) Which of the graphs G_1 to G_6 are plane-graphs? Which of them are planar graphs?

(d) (5 points) On which of G_1 to G_6 are there Eulerian cycles? On which of them are there Eulerian trails? At which vertices are the endpoints in the latter cases?