Elementary Topology Fall 2004
Midterm Exam II (In-Class Part)

The in-class part is worth 60 points, the take-home part 40, altogether 100 points. Theorems from the book and results from exercises we have done can be quoted freely.

1. Write T (True) or F (False) next to each statement. (5 pts each, no partial credit)

(a) In a metric space, the boundary of the open ball $B(x, \varepsilon)$ always contains the set $\{y : d(x, y) = \varepsilon\}$.

(b) Let A be a subset of a topological space X. Suppose A is not empty and not equal to X. If A has empty boundary, then the space is not connected.

(c) Compact sets have nonempty boundary.

2. (a) (5 pts) State the definition of a *metric*.

(b) (5 pts) Given a subset H of a metric space (X, d), how do you use open balls to check whether H is open?

(c) (15 pts) Let (X, d) be a metric space, $z \in X$, and $h \geq 0$. Prove or disprove: the set $H = \{x \in X : d(x, z) > h\}$ is open.

3. (a) (5 pts) Let X be a topological space and $A \subseteq X$. State the definition of the property that A is *compact*.

(b) (15 pts) Let (X, d) be a metric space, and K and L two disjoint compact subsets of X. Show that there exists a positive number $\eta > 0$ such that $d(x, y) \geq \eta$ for all $x \in K$ and $y \in L$.

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Please provide *complete* but *concise*, *polished* solutions to the two questions below. Absolutely no collaboration or consultation with any other person permitted. If you have questions, turn to the instructor. Theorems from the book and results from exercises we have done can be quoted freely.

1. (20 pts) Let

$$H = \prod_{n \in \mathbb{Z}_+} [0, \tfrac{1}{n}],$$

the space of sequences $x = (x_n)_{n \in \mathbb{Z}_+}$ such that $0 \leq x_n \leq \frac{1}{n}$ for each n. Let $\mathcal{T}_p$ be the product topology on H, where each factor $[0, \frac{1}{n}]$ has its standard topology inherited from the real line. Let $\mathcal{T}_u$ be the uniform topology on H defined by the uniform metric

$$\bar{\rho}(x, y) = \sup\{\, |x_n - y_n| : n \in \mathbb{Z}_+\}$$

for sequences $x = (x_n)_{n \in \mathbb{Z}_+}$ and $y = (y_n)_{n \in \mathbb{Z}_+}$ in H.

Determine which of the relations $\mathcal{T}_u \subseteq \mathcal{T}_p$, $\mathcal{T}_p \subseteq \mathcal{T}_u$, $\mathcal{T}_u = \mathcal{T}_p$ are correct and which are false. (H is called the *Hilbert cube*.)

2. (20 pts) Let (X, d) be a compact metric space and f_n a sequence of continuous, nonnegative functions from X into $\mathbb{R}$. Assume each f_n has at least one *zero*, in other words a point $z_n \in X$ such that $f_n(z_n) = 0$. Let f be another continuous nonnegative function from X into $\mathbb{R}$.

(a) Assume that for each $x \in X$, $f_n(x) \to f(x)$ as $n \to \infty$. Does f necessarily have a zero, in other words, does there necessarily exist $z \in X$ such that $f(z) = 0$?

(b) Assume that $f_n \to f$ uniformly. Does f necessarily have a zero, in other words, does there necessarily exist $z \in X$ such that $f(z) = 0$?

(Note: a function f is *nonnegative* if $f(x) \geq 0$ for all x in its domain.)