632 Introduction to Stochastic Processes Spring 2007
Midterm Exam 1

Instructions: Show calculations and give concise justifications for full credit. Points add up to 100.

In-Class Part

1. Consider the Markov chain on the state space $S = \{1, 2, 3, 4\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rows are indexed from top to bottom and columns from left to write, so $p_{1,1} = 1/3$, $p_{2,1} = 1/2$, $p_{3,1} = 1/4$, etc.

(a) (10 pts) Find the transient states, and the closed, irreducible sets of recurrent states.

(b) (15 pts) If the chain starts at state 3, what is the probability that it absorbs eventually in state 4? In other words, find the probability

$$P_3[\text{ there exists a finite number } n_0 \text{ such that } X_n = 4 \text{ for all } n \geq n_0 \text{ }].$$

(c) (20 pts) Find the limits $\lim_{n\to\infty} p_{2,1}^{(n)}$ and $\lim_{n\to\infty} p_{1,2}^{(n)}$. Explain what theorem you are using and what Markov chain are you applying it to.

(d) (15 pts) *Assuming* that the limit $\lim_{n\to\infty} p_{3,1}^{(n)}$ exists, find its value. (We do not have a theorem that guarantees its existence, hence the assumption.) For bonus points, can you give an argument for the existence of this limit?

Take-Home Part

Rules for Take-Home Part: No consultation with anyone permitted. Not with fellow students, not with Internet chat groups, nobody. The Take-Home Part is due by 12 noon tomorrow in the instructor's office.

2. Alice and Betty shoot baskets. Alice starts. She shoots until she misses for the first time. Then the turn passes to Betty, who shoots until she misses for the first time. This way the girls continue, taking turns at the basket, each one shooting until the first miss, at which point the other girl takes a turn. Let us suppose Alice's shots succeed with probability α and Betty's shots succeed with probability β. Each shot succeeds independently of all other shots.

(a) (10 pts) Formulate a Markov chain where the state X_n tells us which girl takes the nth shot, $n = 1, 2, 3, \ldots$

Consider this question: What long term fraction of the shots are taken by Alice? Using your model, formulate the question as a precise mathematical limit statement, appeal to the appropriate theorem(s) from the course material and derive the correct answer.

(b) (10 pts) What is the probability that Alice is the first one to make a successful shot?

3. (20 pts) Ed plays the following game of chance. He starts with one penny. He flips the penny once. If it comes up heads he loses his penny. If it comes up tails he keeps his penny and receives k more pennies. He continues this way. As long as he has at least one penny, he flips a penny, and then either loses it or keeps it and gets k more. Assume all flips are fair (equal chances of heads and tails) and independent of each other. Suppose Ed plays as long as he can and stops only if he goes broke.

Questions: For which values of k is Ed sure to go broke eventually? For which values of k is there some chance that the game goes on forever? In the latter case, is it possible to choose k so that the probability of going broke is below $\frac{1}{2} + \delta$, no matter how small a $\delta > 0$ we specify in advance? Is it even possible to push the probability of going broke down to $\frac{1}{2}$? Be precise about how you model the situation and answer the questions with rigorous reasoning.