632 Introduction to Stochastic Processes Spring 2007
Midterm Exam 2

Instructions: Show calculations and give concise justifications for full credit. Points add up to 100.

In-Class Part

1. Consider a homogeneous rate α Poisson point process on the nonnegative real line $[0, \infty)$. Let $N(A)$ be the number of points in the subset A of $[0, \infty)$.

(a) (10 pts) Given that there are 3 points in $[0, t]$, what is the probability that there are 5 points in $[0, 2t]$? In other words, find the conditional probability

$$P\{N[0, 2t] = 5 \,|\, N[0, t] = 3\}.$$

(b) (10 pts) Given that there are 7 points in $[0, 3t]$, what is the probability that there are 5 points in $[0, t]$? In other words, find the conditional probability

$$P\{N[0, t] = 5 \,|\, N[0, 3t] = 7\}.$$

Now color each point blue with probability p and red with probability $q = 1 - p$. Colors of distinct points are independent.

(c) (10 pts) Let B_3 be the location of the third blue point. Find the expectation $E(B_3)$.

(d) (10 pts) What is the probability that the first 5 points contain at least 4 blue points?

(e) (10 pts) Let X be the location of the third blue point that comes *after* the second red point. (In other words: after the location of the second red point, start counting blue points; the third one is X.) Give some precise description of the probability distribution of X. (Your choice: give the density of X explicitly or as a convolution, or give the Laplace transform of X, or describe X as the result of some arithmetical operation performed on simpler random variables.) Find the expectation $E(X)$.

Take-Home Part

Rules for Take-Home Part: No consultation with anyone permitted. Not with fellow students, not with Internet chat groups, nobody. The Take-Home Part is due by 12 noon tomorrow in the instructor's office.

2. Crabs walk by the cave of the hungry octopus as a homogeneous rate λ Poisson process. When the octopus is not eating a crab, he waits at the opening of his cave, and immediately seizes the first crab that comes by. Then the octopus moves to the far end of his cave to feast on the crab. When the meal is finished the octopus returns to the opening of his cave to wait for the next crab to come by. The octopus is tireless and never fails to catch a crab that presents itself when the octopus is waiting. But crabs that walk by while the octopus is eating go away unharmed.

Assume that it always takes a fixed time c from the moment the octopus catches a crab till he is done with the meal and back at the opening of his cave waiting for the next crab. Let us begin the process when the octopus arrives at the opening of his cave to wait for a crab.

(a) (10 pts) Formulate a regenerative process model for the octopus's life with 2 states: $0 =$ waiting, $1 =$ eating. Be sure to specify the renewal times in your model. Write down the distribution function of the cycle length. (For a more sophisticated answer you can also explain how the forgetfulness property of the exponential distribution is utilized in your model but this is optional.)

(b) (10 pts) Let $R(t)$ denote the number of crabs eaten by time t. Justify the existence of the limit $\lim_{t\to\infty} t^{-1}R(t)$ by appeal to a suitable theorem, and find the value of the limit.

(c) (10 pts) What long term fraction of the crabs fall prey to the octopus?

(d) (10 pts) Suppose you snorkel to the cave at a time t long after the process started. What is the probability that you see the octopus waiting at the opening of the cave? Justify your answer by appeal to a suitable limit theorem.

(e) (10 pts) Suppose you snorkel to the cave at a time t long after the process started. You want to witness the octopus catch a crab. Let $W(t)$ be the time to wait till the next catch:

$$W(t) = \min\{h \geq 0 : \text{the octopus catches a crab at time } t + h\}.$$

Find the limit distribution function

$$F_\infty(x) = \lim_{t\to\infty} P[W(t) \leq x].$$