632 Introduction to Stochastic Processes Spring 2007
Final Exam

Instructions: Show calculations and give **concise justifications for full credit**. Points add up to 200.

1. Born again branching process. Let $\{p_k\}_{0 \le k < \infty}$ be the offspring distribution of a branching process. Assume $0 < p_0 < 1$ so that interesting behavior is possible and let m be the mean of the offspring distribution. Let the process begin with one progenitor, and let X_n be the population size of the nth generation with $X_0 = 1$.

The Galton-Watson branching process we studied in Chapter 1 was absorbed at zero if it ever dies out. Now we give the process a chance to come to life again after extinction. Take another parameter $0 < \beta < 1$. Here is the transition rule for X_n:

(i) If $X_n > 0$ the next state X_{n+1} is determined by the branching process transition with offspring distribution $\{p_k\}$.

(ii) If $X_n = 0$ we put $X_{n+1} = 1$ with probability β and put $X_{n+1} = 0$ with probability $1 - \beta$.

These rules define the transition probability of the Markov chain X_n for all times n.

(a) (10 pts) Can the process X_n be absorbed at 0? Explain.

(b) (20 pts) For what values of m and β is this Markov chain recurrent, for which values transient? Explain.

(c) (20 pts) Let T be the total amount of time the chain X_n spends in state 0, throughout its entire evolution. Compute the mean of T. (The answer should contain as a parameter the extinction probability π of the Galton-Watson branching process with offspring distribution $\{p_k\}$.)

Hint: These questions do not require any special knowledge about branching processes beyond the basic connection between m and the extinction probability. No calculation is necessary for parts (a) and (b).

2. Let α and β be positive constants.

(a) (30 pts) Consider a continuous-time Markov chain $X(t)$ with state space $S = \{0, 1, 2, 3\}$ and jump rates $A(i, i+1) = \alpha$ for $0 \le i \le 2$, $A(j, j-1) = \beta$ for $1 \le j \le 3$. Find the reversible probability distribution for this chain.

(b) (30 pts) Consider a continuous-time Markov chain $X(t)$ that cycles through the state space $S = \{1, 2, 3\}$ with jump rates $A(1, 2) = \alpha$, $A(2, 3) = \beta$, $A(3, 1) = 1$. Is this Markov chain reversible? Find the long term limit probability $\lim_{t\to\infty} P[X(t) = 1]$.

3. (a) (30 pts) Consider a homogeneous rate α Poisson point process on the positive real line $(0, \infty)$. Color the first point red, the second point blue, and continue in this manner to color every other point red, every other point blue. To say it in another way, the first, third, fifth, etc point is red, while the second, fourth, sixth, etc point is blue. Let $W(t)$ be the time till the next blue point:

$$W(t) = \min\{h > 0 : \text{there is a blue point at } t + h\}.$$

Find the limit distribution function

$$F_\infty(x) = \lim_{t\to\infty} P[W(t) \le x],$$

its density $f_\infty(x) = F'_\infty(x)$ and its mean m_∞.

(b) (20 pts) Find the limit

$$\lim_{t\to\infty} P\{\text{the next point after time } t \text{ is blue}\}$$

and justify your reasoning by appeal to a limit theorem.

(c) (30 pts) Contrast part (a) with this situation: now the points of the Poisson rate α process are colored red and blue randomly, with each color equally likely. Let $U(t)$ be the time till the next blue point in this situation. Find the mean $EU(t)$. *Important:* Explain the difference in the means m_∞ and $EU(t)$ by explaining what kind of color distribution the observer sees ahead of her in situation (a) and situation (c).

(d) (10 pts) Let $Y(t)$ be the color of the last point in $(0, t]$ in the process described in (c). To make $Y(t)$ well defined before the first Poisson point T_1, pick a random color (blue or red) and set $Y(t)$ equal to this color for all $t \in [0, T_1)$ You can take it for granted that $Y(t)$ is a Markov chain. Find the rates of this Markov chain to jump from red to blue and blue to red.