632 Introduction to Stochastic Processes Fall 2002
Midterm Exam I

Instructions: Hand in problem no. 1 for 60 points, and *one* other problem for 40 points. Show calculations and justify non-obvious statements for full credit.

General notation: $P_x(A)$ is the probability of the event A when the chain starts in state x, $P_\mu(A)$ the probability when the initial state is random with distribution μ. $T_y = \min\{n \geq 1 : X_n = y\}$ is the first time after 0 that the chain visits y, or ∞ if no visit to y ever happens. $\rho_{x,y} = P_x(T_y < \infty)$ is the probability that the chain visits y some time after time 0, given that it started at x. $N(y)$ is the number of visits to state y, not counting a possible visit at time 0.

1. Fix a constant $\alpha \in (0,1)$, and consider the Markov chain on the state space $S = \{1,2,3,4,5,6\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

(a) Draw the arrow diagram for the Markov chain. Classify the states according to recurrence and transience. Find the periods of recurrent states. (Be careful here because your subsequent answers depend on this part.)

(b) Find the probabilities $\rho_{2,4}$ and $\rho_{6,5}$, and the expectations $E_2N(4)$ and $E_2N(6)$.

(c) Find the probability $P_3[X_3 = 3, X_6 = 3, X_9 = 3, \ldots, X_{3k} = 3]$ for an arbitrary positive integer k.

(d) Find an invariant distribution π.

(e) Find the probability $P_\pi[X_{100} = 4 \text{ or } X_{101} = 4]$. Note that "A or B" means here "A or B or both."

(f) Find $P_\pi[X_{T_4+2} = 4]$, the probability that 2 steps after the first visit to state 4, the chain finds itself again in state 4, and assuming that the initial distribution is π.

2. Consider the success run chain from the homework, with state space $S = \mathbf{Z}_+ = \{0, 1, 2, 3, \ldots\}$ and transition probabilities $p(x, x+1) = \alpha$ and $p(x, 0) = 1 - \alpha$ for all x. Assume $0 < \alpha < 1$. Start the chain at state 1. Find the probability distributions of the random variables T_0 and T_1.

3. Suppose we are talking about an irreducible, recurrent Markov chain with transition probabilities $p(x, y)$ and some arbitrary initial distribution μ. Let u, v and w be 3 distinct states. Let A be the event that v is the next state after the third visit to u (not counting a possible visit to u at time 0). Let B be the event that w is visited some time before the third visit to u. Find the probability $P_\mu(A)$. Show that A and B are independent.

4. Let π be an invariant distribution for a Markov chain X_n with transition probabilities $p(x, y)$, and start the chain with initial distribution π. Fix a time N, and consider the time-reversed process $Y_n = X_{N-n}$, $0 \le n \le N$. Show that

$$P[Y_{n+1} = y | Y_n = x, Y_{n-1} = x_{n-1}, \ldots, Y_0 = x_0] = P[Y_{n+1} = y | Y_n = x]$$

for $0 \le n \le N - 1$. In other words, Y_n is a Markov chain. Find the transition probability for Y_n.

5. Let Y_1, Y_2, Y_3, … be the values in $\{1, 2, 3, 4, 5, 6\}$ of successive rolls of a fair die. Let $\mathbf{u} = (u_1, u_2, \ldots, u_M)$ be an arbitrary fixed M-tuple of numbers from $\{1, 2, 3, 4, 5, 6\}$. Prove or disprove this statement: with probability 1, eventually we see the M-tuple $\mathbf{u}$. In other words, there will be some N (random, but finite) such that $(Y_N, Y_{N+1}, \ldots, Y_{N+M-1}) = (u_1, u_2, \ldots, u_M)$.