632 Introduction to Stochastic Processes Fall 2002
Midterm Exam II

Instructions: Hand in problem 1 for 50 points, problem 2 for 30 points, and **one** other problem for 20 points. Show calculations and justify non-obvious statements for full credit.

1. Fix a constant $\alpha \in (0,1)$, and consider again the Markov chain on the state space $S = \{1,2,3,4,5,6\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1-\alpha & 0 & \alpha & 0 \end{bmatrix}$$

In Exam 1 we found the invariant distribution

$$\pi = \left[\frac{1}{3-\alpha}\ ,\ 0\ ,\ \frac{1-\alpha}{3-\alpha}\ ,\ \frac{1}{3-\alpha}\ ,\ 0\ ,\ 0 \right].$$

(a) State the hypotheses under which we proved the convergence theorem $p^n(x,y) \to \pi(y)$ for Markov chains. Find $\lim_{n\to\infty} p^n(1,3)$ and explain briefly how the hypotheses of the theorem are met.

(b) Find the limiting probability $\lim_{n\to\infty} P_3[X_n = 3, X_{n+3} = 3, X_{n+6} = 3]$.

(c) Starting from 3, find $E_3 T_3$, the expected time to return to state 3.

(d) Suppose the Markov chain receives a reward of 2 dollars every time it visits state 1, but pays a penalty of 3 dollars every time it visits state 3. In the long term, will the Markov chain win money or lose money? How does this answer depend on α? Does the answer depend on where the Markov chain is started?

2. Customers arrive as a homogeneous Poisson process with rate λ.

(a) Given that 2 customers arrive between 8 AM and 9 AM, what is the probability that altogether 6 customers arrive between 8 AM and 12 noon?

(b) Given that 2 customers arrived between 8 AM and 11 AM, what is the probability that the second one came between 9 and 10 AM?

3. Fix constants $\alpha, \beta \in (0, 1)$, and consider the Markov chain on the state space $S = \{1, 2, 3, 4, 5\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \beta & 0 & 1-\beta \\ 0 & 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the limit of $p^n(1, 3)$ as $n \to \infty$.

4. Customers arrive as a homogeneous Poisson process with rate λ per hour. One customer out of 5 is a member of the store club. The amount a member spends has mean 10 and variance 20. Let Y be the total amount spent by member customers during the 10-hour day. Find the mean and variance of Y. State the assumptions required for your answer.

5. Suppose we are looking at a homogeneous Poisson process with rate λ. Given that there are 2 arrivals in the time interval $(0, 4]$, let T be the time of the first arrival. Find the density function of T.