632 Introduction to Stochastic Processes Fall 2003
Midterm Exam I

Instructions: Show calculations and justify non-obvious statements for full credit. When you quote a result proved in class, state the hypotheses and conclusion clearly, and justify why the hypotheses are met. The points add up 100.

General notation: $P_x(A)$ is the probability of the event A when the chain starts in state x, $P_\mu(A)$ the probability when the initial state is random with distribution μ. $T_y = \min\{n \geq 1 : X_n = y\}$ is the first time after 0 that the chain visits y, or ∞ if no visit to y ever happens. $\rho_{x,y} = P_x(T_y < \infty)$ is the probability that the chain visits y some time after time 0, given that it started at x. $N(y)$ is the number of visits to state y, not counting a possible visit at time 0.

1. For this problem, let $\mathbf{P}$ be defined by

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 2/3 & 0 \end{bmatrix}.$$

(a) (15 pts) Explain why the power $\mathbf{P}^n$ of the matrix converges as $n \to \infty$ and find the limit matrix.

(b) (10 pts) Let X_n be a Markov chain with transition matrix $\mathbf{P}$ given above. Let π be the invariant distribution for this chain. Compute the probability

$$P_\pi\big[X_7 = 1, X_{12} = 1\big].$$

(c) (10 pts) Find $P_\pi[X_{T_2+2} = 2]$, the probability that 2 steps after the first visit to state 2, the chain finds itself again in state 2, and assuming that the initial distribution is π.

2. Fix constants $\alpha, \beta \in (0,1)$, and consider the Markov chain on the state space $S = \{1,2,3,4,5\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 0 & \beta & 0 & 1-\beta \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/6 & 5/6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) (10 pts) Draw the arrow diagram for the Markov chain. Classify the states according to recurrence and transience. Find the periods of recurrent states. (Be careful here because your subsequent answers depend on this part.)

(b) (15 pts) Find the probabilities $\rho_{3,4}$ and $\rho_{1,4}$, and the expectations $E_1 N(4)$ and $E_1 N(5)$.

(c) (10 pts) Find the probability $P_1[N(3) = k]$ for all nonnegative integers k, and also for $k = \infty$.

(d) (15 pts) Find *all* invariant distributions π.

(e) (15 pts) Find the limit $\lim_{n\to\infty} p^n(1,3)$.