632 Introduction to Stochastic Processes Spring 2004
Midterm Exam I

Instructions: *Show calculations and give concise justifications for full credit.* Don't forget that theoretical ideas can help avoid tricky computations. The points add up 100.

General notation: $P_x(A)$ is the probability of the event A when the chain starts in state x, $P_\mu(A)$ the probability when the initial state is random with distribution μ. $T_y = \min\{n \geq 1 : X_n = y\}$ is the first time after 0 that the chain visits y, or ∞ if no visit to y ever happens. $\rho_{x,y} = P_x(T_y < \infty)$ is the probability that the chain visits y some time after time 0, given that it started at x. $N(y)$ is the number of visits to state y, not counting a possible visit at time 0.

1. Imagine a road traveled by two kinds of vehicles, trucks and cars. Three out of every four trucks are followed by a car, while one of every five cars is followed by a truck.

(a) (20 pts) Construct a Markov chain model with two states, t for a truck and c for a car, where X_n represents the identity of the nth vehicle coming down the road. Find the invariant distribution of the Markov chain.

(b) (15 pts) Find the limit

$$\lim_{n\to\infty} P_c[X_n = t \text{ or } X_{n+1} = t].$$

In words, this is the long-term probability that two consecutive vehicles contain at least one truck. For the sake of definiteness, we are assuming that the first vehicle we see is a car.

(c) (15 pts) For each integer k, let σ_k be the random index such that X_{σ_k} is the kth truck. Find the limit

$$\lim_{k\to\infty} P_c[X_{\sigma_k+1} = c, X_{\sigma_k+2} = c].$$

In words, this is the probability that the kth truck is followed by two cars.

2. Fix constants $\alpha, \beta \in (0,1)$ such that $\alpha + \beta < 1$, and consider the Markov chain on the state space $S = \{1,2,3,4\}$ with transition matrix

$$\mathbf{P} = \begin{bmatrix} \alpha & \beta & 1-\alpha-\beta & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let τ be the *last time* n such that $X_n = 1$, and if there is no last visit to state 1, set $\tau = \infty$.

(a) (5 pts) Is τ a stopping time? (A mathematical proof is not expected, but give a common sense justification for your answer.)

(b) (25 pts) Find the probabilities $P_1[\tau < \infty]$ and $P_1[X_{\tau+1} = 2]$.

(c) (10 pts) Find the limiting probability

$$\lim_{n \to \infty} P_1[X_n = 2].$$

(It is quicker to use common sense here rather than try to compute.)

3. (10 pts) Start an arbitrary Markov chain at state x. Can it happen that the random variable $N(x)$ is finite with positive probability (in other words, $P_x[N(x) < \infty] > 0$) but still has infinite mean $E_x N(x) = \infty$? Give a short, accurate reasoning that appeals to facts we proved about Markov chains.