Algebra Qualifying Exam
January 2000

Do all **5** problems.

1. Let G be a group of order $2^4 \cdot 5^3 \cdot 11$ and let H be a group of order $5^3 \cdot 11$.
   a. Show that H has a normal Sylow 11-subgroup. (2 points)
   b. If the number of Sylow 5-subgroups of G is (strictly) less than 16, prove that G has a proper normal subgroup of order divisible by 5. (4 points)
   c. If G has exactly sixteen Sylow 5-subgroups, show that G has a normal Sylow 11-subgroup. (4 points)

2. Let R be a (not necessarily commutative) ring with 1 and suppose that R can be written as the sum $R = \sum_{i=1}^{m} I_i$, where the I_i are finitely many (two-sided) ideals of R satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.
   a. Prove that, for every simple right R-module M, there exists a unique subscript k such that $MI_k \neq 0$. (5 points)
   b. Show that if $i \neq j$, then every right R-module homomorphism $\theta\colon I_i \to I_j$ is the zero map. (5 points)

3. Let L/K be a finite degree Galois extension of fields with Galois group given by $\mathrm{Gal}(L/K) = G$, and let E be an intermediate field. Then E is said to be a 2-tower over K if there exists a chain of fields $K = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$ such that $|E_i : E_{i-1}| = 2$ for all $i = 1, 2, \ldots, n$.
   a. If G is abelian, prove that E is a 2-tower over K if and only if the degree $|E : K|$ is a power of 2. (7 points)
   b. Show by example that the characterization of 2-towers given in part (a) is false if G is allowed to be a nonabelian group. (3 points)

4. Let A be an $n \times n$ matrix over the complex numbers and assume that the rank of A is equal to 1.
   a. What are the possible Jordan canonical forms for A? Justify your answer. (5 points)
   b. For each of the forms obtained in part (a), compute the characteristic polynomial of A and the minimal polynomial of A. (5 points)

5. Let $R = F[x, y]$ be the polynomial ring over the field F in the two indeterminates x and y, and let $I = xR$ be the principal ideal of R generated by x. Define $S = F + I$, so that S is a subring of R, and observe that I is an ideal of S.
   a. Show that I is not finitely generated as an ideal of S. (5 points)
   b. Prove that there are infinitely many ideals of S that are not ideals of R. (5 points)