Algebra Qualifying Exam
August 2000

Do all **5** problems.

1. Suppose that a group G is the (internal) direct product of subgroups S and T. Let H be a subgroup of G such that $SH = G = TH$.
   - a. Prove that $S \cap H$ and $T \cap H$ are normal subgroups of G. (4 points)
   - b. If $S \cap H = 1 = T \cap H$, prove that S and T are isomorphic. (3 points)
   - c. If $S \cap H = 1 = T \cap H$ and H is normal in G, show that G is abelian. (3 points)

2. Let $A_1, A_2, \ldots, A_n$ be ideals of the commutative ring R, and let $D = \bigcap_{i=1}^{n} A_i$.
   - a. Prove that $\sqrt{D} = \bigcap_{i=1}^{n} \sqrt{A_i}$. (3 points)
   - b. Now suppose that D is a primary ideal and that it is not the intersection of any proper subset of $\{A_1, A_2, \ldots, A_n\}$. Show that $\sqrt{A_i} = \sqrt{D}$ for all i. (7 points)

3. Let $K \subseteq E$ be a finite degree extension of fields of characteristic 0, and let F_1 and F_2 be intermediate fields. These intermediate fields are said to be *linearly disjoint* over K if $|\langle F_1, F_2\rangle{:}K| = |F_1{:}K||F_2{:}K|$, where $\langle F_1, F_2\rangle$ is the subfield of E generated by F_1 and F_2.
   - a. Prove that $|\langle F_1, F_2\rangle{:}F_1| \leq |F_2{:}K|$ for any F_1 and F_2. (3 points)
   - b. If $|F_1{:}K|$ and $|F_2{:}K|$ are realtively prime, prove that F_1 and F_2 are linearly disjoint over K. (2 points)
   - c. Give an example with $|F_1{:}K| = 2 = |F_2{:}K|$ to show that fields can be linearly disjoint withour having relatively prime degrees. (2 points)
   - d. If F_1 and F_2 are linearly disjoint and Galois over K, prove that the Galois groups satisfy $\mathrm{Gal}(\langle F_1, F_2\rangle/K) \cong \mathrm{Gal}(F_1/K) \times \mathrm{Gal}(F_2/K)$. (3 points)

4. Let V be a complex vector space, not necessarily of finite dimension. Suppose that $A, B\colon V \to V$ are nonzero $\mathbb{C}$-linear transformations with $AB = \lambda BA$ for some fixed nonzero complex number λ. Assume that no proper subspace of V is invariant under both A and V. That is, if W is a subspace of V with $AW \subseteq W$ and $BW \subseteq W$, then $W = 0$ or V.
   - a. Show that A and B are both one-to-one and onto. (5 points)
   - b. If V is finite dimensional, prove that λ is a root of unity. (3 points)
   - c. Show that a finite-dimensional example exists with $\lambda = -1$. (2 points)

5. Let R be a ring and let Z denote its center. A *derivation* $D\colon R \to R$ is a map satisfying $D(a+b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in R$.
   - a. If $r \in R$, show that the map $A_r\colon R \to R$ given by $A_r(a) = ar - ra$, for all $a \in R$, is a derivation of R. (3 points)
   - b. If D is a derivation of R, prove that $D(Z) \subseteq Z$. (3 points)
   - c. If D is a derivation of R and $e \in Z$ is an idempotent, prove that $D(e) = 0$. (Hint. You may need to evaluate $(1-2e)^2$.) (4 points)