Algebra Qualifying Exam
August 2007

Do all **5** problems.

1. Let G be a finite group of order $|G| = 504 = 2^3{\cdot}3^2{\cdot}7$.
   a. If G has a normal subgroup N of order 8, show that G has at most 8 Sylow 7-subgroups, that is $|\mathrm{Syl}_7(G)| \leq 8$. (5 points)
   b. If $|\mathrm{Syl}_7(G)| \leq 8$, prove that G has an element of order 21. (4 points)
   c. If G is isomorphic to a subgroup of Sym_9, the symmetric group of degree 9, show that G cannot have a normal subgroup of order 8. (1 point)

2. Let R be a commutative integral domain with field of fractions F, and assume that R is integrally closed.
   a. Suppose K is a field containing F and let $\alpha \in K$ be integral over R. Show that the minimal monic polynomial of α over F is contained in $R[x]$. (5 points)
   b. Let $f(x) \in R[x]$ be a monic polynomial. Show that $f(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. (5 points)

3. Let F be a field of characteristic 0 and let E be a finite Galois extension of F.
   a. If $0 \neq \alpha \in E$ with $E = F[\alpha]$, show that $F[\alpha^2] \neq E$ if and only if there exists an automorphism $\sigma \in \mathrm{Gal}(E/F)$ with $\alpha^\sigma = -\alpha$. (6 points)
   b. Prove that there exists an element $\alpha \in E$ with $E = F[\alpha^2]$. (4 points)

4. Let V be a finite-dimensional vector space over the field F with $\dim_F V = n$, and let $(\ ,\ ): V \times V \to F$ be a symmetric bilinear form. If X is a subset of V, write $X^\perp = \{v \in V \mid (X, v) = 0\}$ for the subspace of V perpendicular to X.
   a. If W is a subspace of V, show that $\dim_F W + \dim_F W^\perp \geq \dim_F V$. (Hint. If $w \in W$, note that $\{w\}^\perp$ has codimension ≤ 1 in V.) (2 points)
   b. Now suppose $(\ ,\ )$ is nonsingular, so that $V^\perp = 0$. If $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ is a basis for V, prove that there exists a unique dual basis $\mathcal{A}' = \{a_1', a_2', \ldots, a_n'\}$. That is, $\mathcal{A}'$ is a basis with $(a_i, a_j') = 0$ if $i \neq j$ and $(a_i, a_i') = 1$. (4 points)
   c. Again suppose $(\ ,\ )$ is nonsingular, and let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a second basis for V with dual basis $\mathcal{B}' = \{b_1', b_2', \ldots, b_n'\}$. Compare the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$ with the change of basis matrix from $\mathcal{B}'$ to $\mathcal{A}'$. (4 points)

5. Let R be a not necessarily commutative ring with 1.
   a. If $V_1, V_2, \ldots, V_n$ are n nonisomorphic irreducible right R-modules, show that there exists an R-module epimorphism from R, viewed as a right R-module, to the external direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$. (5 points)
   b. Suppose R, viewed as a right R-module, has a finite composition series with nonisomorphic composition factors. Prove that the Jacobson radical of R is equal to 0. (5 points)