Qualifying Examination in Algebra

August, 1978

Instructions:

1. Do four problems.

2. Write each solution in a separate booklet, putting your name and the question number on each booklet.

1. Let G be a finite simple group and let p be a prime dividing $|G|$. Suppose that G has $p+1$ p-Sylow subgroups and that P is a p-Sylow subgroup. Show that

(i) G is isomorphic to a subgroup of the symmetric group of degree $p+1$. (3 points)

(ii) p^2 does not divide $|G|$. (2 points)

(iii) The centralizer of P in G is equal to P. (2 points)

(iv) $|G|$ divides $p(p^2-1)$. (3 points)

2. Let G be a finite group and let $N_1, N_2, \ldots, N_k$ be normal subgroups. Suppose that $\cap_1^k N_i = \langle 1 \rangle$, and that $G/N_i = S_i$ is simple. If all the groups S_i are non-isomorphic prove that

$$G \simeq S_1 \times S_2 \times \ldots \times S_k$$ (10 points)

3. Let R be a ring with 1 and let V be a left R-module satisfying the ascending and descending chain conditions on submodules. If $T: V \to V$ is an R-endomorphism of V prove that there is an integer m such that

$$(\ker T^m) + (\operatorname{im} T^m) = V$$

and

$$(\ker T^m) \cap (\operatorname{im} T^m) = 0$$ (10 points)

4. Let L_1 and L_2 be minimal left ideals of a ring R (with 1).

(i) Show that if the product $L_1 L_2$ is nonzero then $L_1 \simeq L_2$ as R-modules. (5 points)

(ii) Conversely, if R is semisimple Artinian and $L_1 \simeq L_2$ as R-modules show that $L_1 L_2$ is nonzero. (5 points)

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. Let $R = \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients. Let $I = (4, x)$ be the ideal of R generated by 4 and x.

(i) Compute the radical of I. (3 points)

(ii) Prove that I is primary. (4 points)

(iii) Prove that I is not a power of its associated prime. (3 points)

6. Let F be a field, R a subring with 1 and let α be a nonzero element of F. Suppose that

$$R[\alpha] = F = R[1/\alpha] .$$

(i) Show that F is integral over R. (5 points)

(ii) Show that R is a field. (5 points)

7. Let $\mathbb{Q}$ be the rational field and let ζ be a primitive fifth root of unity. The field $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$ of degree 4.

Prove that $\mathbb{Q}(\zeta)$ has a unique subfield K of degree 2 over $\mathbb{Q}$ (4 points) and that $K = \mathbb{Q}(\sqrt{5})$ (6 points).

8. Let K_i $(i = 1, 2)$ be Galois extensions of a field k with Galois groups G_i. Assume that K_1, K_2 are included in some field and let $K_3 = K_1 K_2$ be their composite.

(i) Show that K_3 is a Galois extension of k. (4 points)

(ii) Show that if $n_1 = [K_1 : k]$ and $n_2 = [K_2 : k]$ are relatively prime integers then the Galois group G_3 of K_3/k is isomorphic to $G_1 \times G_2$. (6 points)