QUALIFYING EXAM IN ALGEBRA

August 1979

Do four problems.
Write each solution in a separate booklet.
Write your name on each booklet.

1. Let p be a prime and let G be a finite group with the property (*) that every element of order a power of p is contained in a conjugacy class of size a power of p.

   i) If $p \mid |G|$, show that G has a central subgroup Z of order p. (4 points)

   ii) If Z is as above, show that G/Z has property (*). (4 points)

   iii) Deduce that a Sylow p-subgroup of G is normal. (2 points)

2. Let G be a finite group with the property that all of its proper subgroups are abelian. Let N be a proper normal subgroup of G. Show that either N is contained in the center of G or else G has a normal abelian subgroup of prime index.

3. Recall that an R-module X is said to be completely reducible if every submodule is a direct summand or, equivalently, if X is a (possibly infinite) direct sum of simple submodules. Let M be an R-module and let K be a submodule such that M/K is completely reducible. Suppose $N \subseteq M$ and $N \cap K = 0$. Prove that

   i) N is a direct summand of M. (5 points)

   ii) N is Artinian iff it is Noetherian. (HINT: Show that N is completely reducible.) (5 points)

4. In this problem we consider commutative rings with 1.

   i) Suppose every ideal $I \neq R$ of a ring R is prime. Prove that R is a field. (HINT: Consider the ideal (x^2) for $x \in R$.) (4 points)

   ii) Suppose that every nonzero proper ideal of a ring is maximal. Prove that there are at most two such ideals. (4 points)

   iii) Suppose that I is an ideal of a ring R which is maximal with respect to the property that it is proper and not prime. Deduce that I is contained in at most two other proper ideals of R. (HINT: Apply parts (i) and (ii) in turn to suitable factor rings of R.) (2 points)

(MORE ON BACK)

Let V be an infinite dimensional vectorspace over a field F and let R be the ring of all F-linear transformations from V to V. Note that this makes V into a simple right R-module. Let S be the set of all $r \in R$ with finite dimensional range, and let I be any nonzero 2-sided ideal of R.

i) Show that S is a 2-sided ideal of R. (3 points)

ii) If $v \in V$ with $v \neq 0$, show that $v \cdot I = V$. (3 points)

iii) If $r \in R$ has 1-dimensional range, prove that $r \in rI \subseteq I$. (2 points)

iv) Deduce that $S \subseteq I$. (2 points)

6. Let F be a field and suppose $\alpha \in F$. Let m and n be relatively prime positive integers.

i) Let $K \supseteq F$ be a field. Show that K contains an (mn)-th root of α iff it contains both an m-th root and an n-th of α. (5 points)

ii) Prove that the polynomial $x^{mn} - \alpha$ is irreducible over F iff both $x^m - \alpha$ and $x^n - \alpha$ are irreducible over F. (HINT: Use field extension degrees.) (5 points)

7. Let $E \supseteq F$ be a finite degree, Galois field extension. Let $f(x) \in F[x]$ be an irreducible polynomial of prime degree. Suppose $f(x)$ reduces over E. Show that $f(x)$ splits completely in $E[x]$.

8. Let $E \supseteq F$ be a finite degree, Galois field extension. Assume that all intermediate fields, properly between E and F, have equal degrees over F. Prove that all of these intermediate fields are Galois extensions of F.