# Qualifying Exam in Algebra

January 1982

Do FOUR problems.

1. Let $P$ be a nonabelian group of order $p^4$, where $p$ is a prime, and let $A$ be a subgroup of $P$ maximal with the property of being normal and abelian. Show that $|A| = p^3$. HINT: Consider the centralizer of $A$.

2. Let $K = Q[\zeta]$ where $Q$ is the field of rational numbers and $\zeta$ is the complex number given by $\zeta = e^{2\pi i/63}$.

   a) How many subfields of $K$ are extensions of degree 2 or 3 over $Q$? (5 points)

   b) How many of the subfields counted in part (a) are contained in the real numbers? (5 points)

   Justify your answers!

3. Let $F$ be a field and let $R = F[x,y,z]$ be the polynomial ring in the three indeterminates $x,y,z$ over $F$. Let $I$ be the ideal of $R$ generated by $xy$, $xz$ and $yz$. Show that $I$ is an intersection of three prime ideals of $R$.

4. Let $V$ be a vector space over the reals with a positive definite symmetric bilinear form $( \, , \, )$. Let $v \in V$ be some fixed vector. Suppose $u_1, u_2, \cdots, u_n \in V$ are vectors such that

i) $(u_i, v) > 0$ for all $i$

ii) $(u_i, u_j) \leq 0$ for all $i \neq j$.

Show that $u_1, u_2, \cdots, u_n$ are linearly independent.

5. A group $G$ is said to be complete if

i) all its automorphisms are inner, and

ii) its center is trivial.

a) Show that Sym(3), the symmetric group on three symbols, is complete. (5 points)

b) Suppose a complete group $G$ is contained as a normal subgroup in some group $X$. Show that $G$ is a direct factor of $X$. (5 points)

6. Let $K$ be a field of characteristic zero which contains a primitive $n^{th}$ root of unity $\varepsilon$. Suppose that $a \in K$ and that the polynomial $f(x) = x^{2n} + ax^n + 1$ is irreducible in $K[x]$. Let $L$ be a splitting field for $f(x)$ over $K$.

a) If $\alpha \in L$ is a root of $f(x)$, find the remaining roots. (3 points)

b) Determine the degree $|L:K|$. (3 points)

c) Find the number of elements of order $2$ in the Galois group $Gal(L/K)$. (4 points)

HINT for (c): Consider separately the cases $n$ odd and $n$ even.

7. Let $R$ be a (not necessarily commutative) ring with $1$, and let $e$ be a nonzero idempotent in $R$. Let $J = J(R)$ be the Jacobson radical of $R$.

a) Show that the ideal $eJe$ of the ring $eRe$ is quasi-regular. (3 points)

b) If $V$ is a simple right $R$-module, show that either $Ve = 0$ or $Ve$ is a simple right $eRe$-module. (4 points)

c) Conclude that $eJe$ is the Jacobson radical of $eRe$. (3 points)

8. Let $G$ be the group defined by the presentation

$$G = \mathrm{gp}\langle a,b,x,t \mid a^2 = b^2 = 1 \quad [a,b] = 1$$
$$x^3 = 1 \quad a^x = b \quad b^x = ab$$
$$t^2 = 1 \quad x^t = x^{-1} \quad a^t = a \quad b^t = ab \rangle .$$

Prove that $G \cong \mathrm{Sym}(4)$, the symmetric group on four symbols.

## Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.