QUALIFYING EXAM IN ALGEBRA

August, 1982

Do FOUR problems.

1) Let G be a finite group and let P be a subgroup of prime order p. Assume that $P = \mathbb{C}_G(P)$, that is P is self-centralizing.

a) Show that p^2 does not divide $|G|$. (3 pts)

b) If $N \lhd G$ and $P \not\subseteq N$, show that p divides $|N| - 1$. (3 pts)

c) If $N \lhd G$ and $P \subseteq N$, show that the index $|G:N|$ divides $p-1$. (4 pts)

2) Let G be a finite group and let H be a solvable subgroup of G. Let $N = \mathbb{N}_G(H)$ be the normalizer of H in G and assume that N/H is a nonabelian simple group. Prove that $N = \mathbb{N}_G(N)$.

3) Let R be a commutative Noetherian ring with 1 and let M be a nonzero right R-module. Let S be the set of all ideals I of R such that $mI = 0$ for some nonzero $m \in M$.

a) Show that the maximal members of S (with respect to inclusion) are prime ideals. (4 pts)

b) For each maximal member P of S let $U(P) = \operatorname{ann}_M(P) = \{m \in M \mid mP = 0\}$. Show that the sum of these submodules U(P) is direct. (4 pts)

c) If M is a finitely generated R-module, deduce that S contains only finitely many maximal members. (2 pts)

4) Let R be a ring with 1 which has only finitely many units.

a) Let J be the Jacobson radical of R. Prove that J is finite and that R/J has only finitely many units. (5 pts)

b) If R is commutative and infinite, prove that R has infinitely many maximal ideals. (5 pts)

5) Let $F = \mathbb{Z}/7\mathbb{Z}$ be the field of integers modulo 7. Let $I = (x^2-x-1)$ and $J = (x^2+1)$ be the ideals of the polynomial ring $F[x]$ generated by the irreducible polynomials x^2-x-1 and x^2+1. Since $F[x]/I$ and $F[x]/J$ are finite fields with the same number of elements, they are isomorphic. Find an explicit isomorphism $F[x]/J \to F[x]/I$ and prove it is an isomorphism.

6) Let $\theta = \sqrt{5 + \sqrt{5}}$. Show that $\mathbb{Q}[\theta]$ is a Galois extension of the rationals $\mathbb{Q}$ and determine the Galois group $\mathrm{Gal}(\mathbb{Q}[\theta]/\mathbb{Q})$ as an explicit group of permutations of the roots of the minimal polynomial of θ over $\mathbb{Q}$.

7) Let A be an $n \times n$ complex matrix. Prove that the following statements are equivalent.

a) $\operatorname{rank} A^2 = \operatorname{rank} A$

b) The multiplicity of 0 as a root of the minimum polynomial of A is at most one.

c) There is an $n \times n$ matrix X such that

$$AXA = A \;, \quad XAX = X \;, \quad AX = XA \;.$$

8) Let $G = GL(6,2)$, the group of invertible 6×6 matrices over the field $\mathbb{Z}/2\mathbb{Z}$. Let E be the set of all matrices in G of the form

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$$

where I is the 3×3 identity matrix, 0 is the 3×3 zero matrix and X is any 3×3 matrix over $\mathbb{Z}/2\mathbb{Z}$.

a) Prove that E is an elementary abelian 2-subgroup of G. (3 pts)

b) Show that $\mathbb{C}_G(E) = E$, that is E is self-centralizing in G. (3 pts)

c) If $N = \mathbb{N}_G(E)$ is the normalizer in G of E, prove that $N/E \simeq GL(3,2) \times GL(3,2)$. (4 pts)