QUALIFYING EXAM

ALGEBRA

January 17, 1983

Do FOUR problems.

1) Let T be a linear transformation on a finite dimensional complex vector space, V. Suppose that the subspace of V spanned by all eigenvalues of T is 1-dimensional. Prove that the minimal polynomial of T is equal to $(x-\alpha)^n$, where $n = \dim V$ and α is some complex number.

2) Let R be a right Artinian ring with 1. Suppose that R has only finitely many units (i.e. elements with 2-sided inverses). Show that R itself has only finitely many elements.

3) Let S be the set of all complex numbers which are algebraic over the rationals $\mathbb{Q}$ and which have a minimal polynomial over $\mathbb{Q}$ that is solvable by radicals.

a) (5 pts.) Show that S is a field.

b) (5 pts.) Show that, in the complex numbers, S is closed under taking nth roots (for arbitrary positive integers n). Show also that S is minimal with this property.

4) The socle of a finite group G is, by definition, the subgroup generated by all nonidentity minimal normal subgroups of G. This characteristic subgroup is denoted $soc(G)$.

a) (4 pts) Suppose N is a normal subgroup of G with $soc(G) \subseteq N \lhd G$. Show that $soc(G) \subseteq soc(N)$.

b) (3 pts) Suppose G has no nontrivial abelian normal subgroup and show that then the centralizer $\mathbb{C}_G(soc(G)) = 1$.

c) (3 pts) Suppose $soc(G) \subseteq N \lhd G$ and that G is as in part b). Show that then $soc(G) = soc(N)$.

5) Let R be an integral domain with quotient field K and suppose that R is integrally closed in K (that is, all elements of K which are integral over R are in R). Let F be an algebraic field extension of K.

a) (5 pts) Suppose that $\alpha \in F$ is a root of a monic polynomial in $R[x]$. Show that the minimal polynomial for α over K has coefficients in R.

b) (5 pts) Show that there is a subring S of F such that F is the field of quotients of S and $S \cap K = R$.

6) Let G be a finite group which has exactly eight Sylow 7-subgroups. Show that there exists a normal subgroup N of G such that the index $|G:N|$ is divisible by 56 but not by 49.

7) a) (3 pts) Determine the minimal polynomial, $p(x)$, of $\sqrt{2+\sqrt{2}}$ over the rationals, $\mathbb{Q}$.

b) (3 pts) Determine the degree of the splitting field F for $p(x)$ over $\mathbb{Q}$ in $\mathbb{C}$, the complex numbers.

c) (4 pts) Determine the Galois group of F over $\mathbb{Q}$.

8) Let $R \subseteq S$ be commutative rings with the same 1 . Let P be a minimal prime ideal of R (i.e. P contains no properly smaller prime ideal of R , in particular, $P = (0)$ if (0) is a prime ideal). Show that there exists a prime ideal Q of S with $Q \cap R = P$.

POLICY ON MISPRINTS

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.