Qualifying Exam

ALGEBRA

January 16, 1984

Instructions: Do four problems.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1. The socle soc(G) of a finite group G is the subgroup generated by all minimal normal subgroups of G.

   a) Let $S = \mathrm{soc}(G)$. Prove that $S = \mathrm{soc}(S)$. (5 points)

   b) If $G = \mathrm{soc}(G)$ and M is minimal normal in G, prove that M is simple. (5 points)

2. Let $F \subseteq E$ be a finite degree field extension. Suppose for each intermediate field K such that E is primitive (i.e. singly generated) over K, that E is separable over K. Show that E is separable over F.

3. Let R be a commutative ring with 1 and let $A_1, A_2, \ldots, A_n$ be ideals of R. Set $B = \bigcap_{i=1}^{n} A_i$.

a) Suppose that $A_i + A_j = R$ for all $i \neq j$. Prove that R/B is isomorphic to the (external) direct sum of the rings R/A_i. (7 points)

b) Show by example that the above isomorphism does not hold in general if the assumption $A_i + A_j = R$ (for all $i \neq j$) is replaced by the weaker condition $A_1 + A_2 + \ldots + A_n = R$. (3 points)

4. Let p be a prime, let q be a power of p and set $\omega = e^{2\pi i/q}$. Suppose $c_0, c_1, \ldots, c_{q-1} \in \mathbb{Z}$ (the integers) and that

$$\sum_{i=0}^{q-1} c_i \omega^i \in \mathbb{Z} .$$

Prove that

$$\sum_{i=0}^{q-1} c_i \omega^i \equiv \sum_{i=0}^{q-1} c_i \mod p .$$

5. A group X is said to be <u>perfect</u> if $X = X'$, where X' is the commutator subgroup of X. Let $\varphi : G \to H$ be a homomorphism from the finite group G onto the perfect group H.

a) Show that there exists a normal, perfect subgroup N of G with $\varphi(N) = H$. (5 points)

b) Assume that the kernel of φ is solvable. Show that the subgroup N of part (a) is unique. (5 points)

6. Let $f(x) = x^5 - 6x + 2$ be a polynomial over the rationals Q. Let $\alpha_1, \alpha_2, \ldots, \alpha_5$ be its roots in the complex numbers and set $K = Q[\alpha_1, \alpha_2, \ldots, \alpha_5]$.

a) Show that the degree $|K:Q| = 120$. (3 points)

b) Let $g(x) = \prod_{1 \leq i < j \leq 5} (x - \alpha_i \alpha_j)$. Show that $g(x) \in Q[x]$ and that $g(x)$ is irreducible over Q. (5 points)

c) Determine the splitting field of $g(x)$ in the complex numbers. (2 points)

7. Let R be a right Artinian ring with 1 and let $J = J(R)$ be its Jacobson radical. Let I be a 2-sided ideal of R.

a) Suppose (for this part only) that $I \cap J = 0$. Prove that there exists a 2-sided ideal K of R with $R = I \oplus K$. (5 points)

b) Show that every unit of R/I is of the form $u + I$ where u is a unit of R. (Hint. Consider the cases $I \cap J = 0$ and $I \subseteq J$). (5 points)

8. An abelian group D (written additively) is said to be <u>divisible</u> if, for every integer n and element $d \in D$, there exists $x \in D$ with $nx = d$. Let $D \subseteq A$ be additive abelian groups with D divisible.

a) If $D \neq A$, show that there exists a subgroup $B \subseteq A$ with $B \neq 0$ and $B \cap D = 0$. (5 points)

b) Show that D is a direct summand of A. (5 points)