3/

Qualifying Exam

ALGEBRA

August 27, 1984

Instructions: Do four problems.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

Algebra Q.E.
August 27, 1984
Page 1

1. Let G be a finite group and let P be a subgroup of prime order p. Suppose P is equal to its centralizer $\mathbb{C}_G(P)$ in G.

   a. Show that P is a Sylow p-subgroup of G. (3 points)

   b. If P is normal in G, show that G is solvable. (3 points)

   c. Let $q \neq p$ be a prime divisor of $|G|$. If P is normal in G, find the number of elements of G of order q. Prove your answer. (4 points)

2. Definition. Let S be a (not necessarily commutative) ring with 1. A proper ideal P of S is prime if whenever P contains the product AB of two ideals of S then $P \supseteq A$ or $P \supseteq B$.

   Let R be a ring and e a nonzero idempotent element of R. Note that eRe is a ring with identity e.

   a. If I is an ideal of R, show that eIe is an ideal of the ring eRe. Furthermore if I is prime in R and $e \notin I$ show that eIe is prime in eRe. (5 points)

   b. If J is an ideal of the ring eRe, show that there exists a unique largest ideal I of R such that $eIe = J$. Furthermore if J is prime in eRe, show that I is prime in R. (5 points)

3. Let F be a field with algebraic closure $\tilde{F}$. Suppose that for each integer n there exists at most one intermediate field L such that the degree $(L:F) = n$. Let $F \subseteq K \subseteq \tilde{F}$ with $(K:F) < \infty$.

   a. Prove that K is the splitting field of an irreducible polynomial over F. (5 points)

   b. If $G = \mathrm{Gal}\,(K/F)$, show that G is cyclic. (5 points)

4. Let A be the (complex) matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$$

   a. Find the minimum polynomial of A. (3 points)

   b. Find the Jordan canonical form J of A. (3 points)

   c. Find an invertible matrix P such that $P^{-1}AP = J$. (4 points)

Justify your answers.

5. Let G be a finite solvable group and let $x_1, x_2, \ldots, x_n$ be elements of G of pairwise relatively prime orders. If the product $x_1 x_2 \ldots x_n = 1$, show that each $x_i = 1$.

6. Let R be a commutative Noetherian ring with 1 and let $R[x]$ be the polynomial ring in the indeterminate x. Let I be an ideal of R and define

$$S = R + Ix + I^2x^2 + \cdots + I^nx^n + \cdots \subseteq R[x] \quad .$$

a. Show that S is a subring of $R[x]$ and that S is a homomorphic image of a polynomial ring over R in finitely many indeterminates. (5 points)

b. Let J be an ideal of R. For each integer $n \geq 0$ let

$$J_n = J + (I \cap J)x + (I^2 \cap J)x^2 + \cdots + (I^n \cap J)x^n \subseteq S$$

and set

$$\tilde{J} = \bigcup_{n=0}^{\infty} J_n \quad .$$

Show that $\tilde{J}$ is an ideal of S and that $\tilde{J} = J_nS$ for some n. (5 points)

7. Let Q be the field of rational numbers, let C be the complex field and let $\alpha \in C$.

   a. Let $Q \subseteq E \subseteq C$ with E a (finite) Galois extension of Q. Suppose U is a field with $Q(\alpha) \subseteq U \subseteq E(\alpha)$. Show that $U = K(\alpha)$ where $K = U \cap E$. (Hint. Compute the degrees $(E(\alpha):U)$ and $(E(\alpha):K(\alpha))$). (5 points)

   b. Assume α is transcendental over Q and let

   $$\beta = \sqrt{2} + \sqrt{3}\,\alpha + \sqrt{5}\,\alpha^2 .$$

   Show that $\sqrt{2},\sqrt{3},\sqrt{5} \in Q(\alpha,\beta)$. (Hint. In part (a) choose E to contain $\sqrt{2},\sqrt{3},\sqrt{5}$ and deduce that $\beta \in K(\alpha)$. Now write β as $p(\alpha)/q(\alpha)$ where $p(x),q(x) \in K[x]$.) (5 points)

8. Let A and B be $m \times n$ matrices over a field F.

   a. Prove that:

   $$\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B) .$$ (4 points)

   b. If equality occurs in part (a), show that every row vector v in F^m can be written as a sum $v = u + w$ of row vectors with $uA = 0$ and $wB = 0$.

   (6 points)