Qualifying Exam

ALGEBRA

January 14, 1985

Instructions: Do four problems. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

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1. Let G be a transitive permutation group of prime degree p. (Note that this implies that every nonidentity normal subgroup of G is also transitive.) Assume that G is solvable.

   a) Show that $G = PH$ where $P \lhd G$ has order p, H is abelian and $P \cap H = 1$. (5 points)

   b) If G is doubly transitive, find the order of H. (3 points)

   c) If G is triply transitive, show that $p = 3$. (2 points)

2. Let R be a commutative ring with 1 and suppose Q is a primary ideal of R. If the radical of Q is a principal ideal, show that Q is also a principal ideal.

3. Let $f(x)$ be a nonconstant polynomial over the rationals Q and let G be the Galois group of f over Q. For each integer $k \geq 0$ set $f_k(x) = f(x^{2^k})$ and let G_k be the Galois group of $f_k(x)$ over Q.

   a) Show that there is a homomorphism from G_1 onto $G = G_0$ with kernel which is an elementary abelian 2-group. (6 points)

   b) Show that for some k, there is a homomorphism of G_k onto G with kernel a nontrivial elementary abelian 2-group. (4 points)

4. Let F be the field of 3 elements and let V be a 2-dimensional vector space over F. Let G be the group of all invertible linear transformations of V and let Z be its center. Show that $G/Z \simeq \mathrm{Sym}(4)$, the symmetric group on 4 letters.

5. Let G be a finite group and let X and L be normal subgroups of G. If P is a Sylow p-subgroup of L, show that the normalizer of PX satisfies

$$\mathbb{N}_G(PX) = \mathbb{N}_G(P) \cdot (X \cap L) \quad .$$

6. Let R be a <u>right</u> Artinian ring with 1 and let J be its Jacobson radical. Let $e \neq 0$ be an idempotent of R and assume that eR is a simple right R-module.

   a) Show that Re/Je is a simple left R-module. (7 points)

   b) Give an example to show that Re need not be a simple left R-module. (Hint. Consider a ring of 2×2 upper triangular matrices.) (3 points)

7. Let F be a field of prime characteristic p and let K be an algebraic extension of F. Let S be the subfield of K consisting of all elements $\alpha \in K$ such that $\alpha^{p^n} \in F$ for some integer $n \geq 0$ (depending on α).

   a) Suppose both $\beta, \gamma \in K$ are roots of the irreducible polynomial $f(x)$ over F. Show that they are both roots of the same irreducible polynomial $g(x)$ over S. (5 points)

   b) If K is normal over F, show that K is separable over S. (5 points)

8. Let V be an n-dimensional vector space over a finite field of q elements and let $1 \leq k \leq n$.

   a) Find the number of subspaces of dimension exactly k. (5 points)

   b) Find the number of linear transformations $T : V \to V$ of rank exactly k. (5 points)

   Prove your answers in both parts.