Qualifying Exam

ALGEBRA

August 25, 1986

Instructions: Do FOUR questions.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

1. Let G be a finite group with normal subgroup N and assume that G/N is nilpotent.

i. Show that G has a nilpotent subgroup K such that $G = NK$. (5 pts)

ii. If N is abelian and G has trivial center, prove that any nilpotent subgroup K of G with $G = NK$ satisfies $K \cap N = 1$ and $K = \mathbf{N}_G(K)$. (5 pts)

2. i. Let R be a commutative ring with 1 having exactly one maximal ideal M. Let $a,b \in R$ and suppose that the principal ideals (a) and (b) are equal. Show that $a = bu$ for some unit $u \in R$. (5 pts)

ii. Let $\mathbf{Z}$ be the ring of integers and let $p > 3$ be a prime. Let R be the subring of $\mathbf{Z} \oplus (\mathbf{Z}/p^2\mathbf{Z})$ consisting of those elements $(u,\bar{v})$ such that $u \equiv v \mod p$. Consider the elements $a = (0,\overline{2p})$ and $b = (0,\bar{p})$ in R. Show that $(a) = (b)$ but that there is no unit u of R with $a = bu$. (5 pts)

3. Let K be a field and let $f(x) \in K[x]$ be a separable polynomial. Let E be a splitting field for $f(x)$ over K and let F be the subfield of E generated over K by all elements $\alpha-\beta$ where α and β are roots of $f(x)$ in E.

i. Prove that the degree $(E:F)$ equals 1 if char $K = 0$ and is a power of p if char $K = p > 0$. (Hint. Determine the order of a field automorphism of E over F.) (8 pts)

ii. Give an example which shows that E need not equal F. (2 pts)

4. Let V be a complex vector space with basis $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ where $n \geq 5$. Let G be the multiplicative group of those linear transformations of V whose matrix with respect to the basis $\mathcal{B}$ has precisely one nonzero entry in each row and column, and where each such nonzero entry is either $+1$ or -1.

i. Find the order of G and list the factors in some composition series. (5 pts)

ii. Let W be a nonzero subspace of V, which is mapped into itself by each element of G. Prove that $W = V$. (5 pts)

5. Let G be a finite simple group.

i. If G has a proper subgroup of index ≤ 9, show that G has no elements of order 21. (3 pts)

ii. If $|G| = 504$, find the number of Sylow 7-subgroups of G. Prove your answer. (4 pts)

iii. Again assume that $|G| = 504$. Show that G has no elements of order 21. (3 pts)

6. Let R be a commutative ring with 1 and let S be a subring with the same 1.

i. Assume that R is the direct sum $S \dotplus T$ where T is an S-submodule of R. If N is a maximal ideal of S, show that $N = S \cap M$ for some maximal ideal M of R. (5 pts)

ii. Show by example that (i) can fail if we do not assume the existence of T. (1 pt)

iii. Now suppose $R = K[x_1, x_2, \dots, x_n]$ is a polynomial ring over the field K in n variables and that S is a subring containing K. If M is a maximal ideal of R, prove that $S \cap M$ is maximal in S. (4 pts)

7. We say that a field extension $T \supseteq F$ is purely transcendental if the only elements of T algebraic over F are in F. Let $E \supseteq T \supseteq F$ be fields with T purely transcendental over F.

i. Let $\alpha \in E$ be algebraic over F with minimal polynomial $f(x) \in F[x]$. Show that $f(x)$ is irreducible in $T[x]$. (4 pts)

ii. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$ be separable algebraic elements over F which are linearly independent over F. Show that they are linearly independent over T. (6 pts)

8. Let $K \subseteq E$ be fields. Let $\delta: E \to E$ be a K-linear map satisfying

$$\delta(ab) = a\,\delta(b) + \delta(a)\,b$$

for all $a, b \in E$. (Such a map is called a K-derivation.)

i. Show that $\delta(k) = 0$ for all $k \in K$. (2 pts)

ii. Let $f(x) \in K[x]$ and let $f'(x)$ be its formal derivative. Show that $\delta(f(a)) = f'(a)\,\delta(a)$ for all $a \in E$. (3 pts)

iii. If E/K is algebraic and separable, prove that $\delta = 0$. (5 pts)