# ALGEBRA QUALIFYING EXAM August 1987

1. Let $G$ be a finite group and let $P \in Syl_p(G)$ be a Sylow $p$-subgroup of $G$. Let $Q$ be a characteristic subgroup of $P$.

i. Show that the number of conjugates of $Q$ in $G$ is congruent to $1$ mod $p$. (5 points)

ii. Show that the number of these conjugates which happen to be contained in $P$ is also congruent to $1$ mod $p$. (5 points)

2. Let $R$ be a commutative Noetherian ring with $1$.

i. Let $A$ and $B$ be ideals of $R$ such that $R/A$ and $R/B$ are finite. Prove that $R/(AB)$ is finite. (5 points)

ii. Assume that $R/P$ is finite for every prime ideal $P$ of $R$. Show that $R$ is finite. (5 points)

3. Let $F \subsetneq E$ be a finite degree Galois extension with Galois group $G$.

i. Suppose $E$ is the splitting field of the irreducible polynomial $f(x) \in F[x]$ and let $e \in E$ be a root of $f(x)$. If $H$ is the subgroup of $G$ given by $H = \{ \sigma \in G \mid \sigma(e) = e \}$, show that $H$ contains no nonidentity normal subgroup of $G$. (5 points)

ii. Now let $H$ be any subgroup of $G$ which contains no nonidentity normal subgroup of $G$. Show that $E$ is the splitting field of an irreducible polynomial $f(x) \in F[x]$ with $\deg f(x)$ equal to the index $|G : H|$. (5 points)

4. Let $K$ be a field and fix an integer $n \geq 2$. Let $M$ be the ring of $n \times n$ matrices over $K$ and let $S \subseteq M$ be the subset consisting of all matrices of trace 0. Assume that either $n \geq 3$ or $n = 2$ and the characteristic of $K$ is not 2. Prove that every element of $S$ is a $K$-linear combination of matrices of the form $\alpha\beta - \beta\alpha$ with $\alpha, \beta \in S$.

5. Let $G$ be a finite group and let $N$ be a normal subgroup of $G$.

i. If $N$ is nilpotent and $M$ is a maximal subgroup of $G$, show that $M \cap N$ is normal in $G$. (5 points)

ii. Conversely suppose that $M \cap N$ is normal in $G$ for every maximal subgroup $M$ of $G$. Prove that $N$ is nilpotent. (5 points)

6. Let $R$ be a ring with 1 and let $J(R)$ be its Jacobson radical. Suppose $S$ is a subring of $R$ such that $R = S + J(R)$.

i. Let $1_S$ be the identity element of $S$. Show that $1_S = 1$. (3 points)

ii. Prove that $J(S) \subseteq J(R) \cap S$. (4 points)

iii. If $R$ is right Artinian deduce that $J(S) = J(R) \cap S$. (3 points)

7. Let $F$ be a field of prime characteristic $p$ and let $f(x) \in F[x]$ be a polynomial of degree $n$. (Recall that a polynomial in $F[x]$ is *separable* if no irreducible factor has repeated roots.)

i. If $p \nmid n$ show that there are at most finitely many elements $c \in F$ such that the polynomial $f(x) + c$ is not separable. (5 points)

ii. Give an example (with proof) where $p \mid n$ and $f(x) + c$ is not separable for infinitely many values $c \in F$. (5 points)

8. Let $\mathcal{Q}$ be a collection of pairwise commuting $n \times n$ matrices over the complex numbers $\mathbb{C}$. Prove that there exists a vector which is a common eigenvector for all matrices in $\mathcal{Q}$.