ALGEBRA QUALIFYING EXAM
August, 1988

Instructions: Do precisely 4 of the following 8 problems.

1. Let G be a finite group and suppose $G = AB$ where A and B are normal subgroups of G with $A \cap B = 1$. Show that $|A|$ and $|B|$ are coprime (that is, relatively prime) if and only if for every subgroup $H \subseteq G$ we have $H = (H \cap A)(H \cap B)$.

2. a. Let R be a ring with 1 and let M be a right R-module. If the endomorphism ring $\mathrm{End}_R(M)$ is infinite, prove that the module $M \oplus M$ has infinitely many submodules. (5 points)

b. Let R be a simple ring with 1 having only finitely many right ideals. Show that either R is finite or it is a division ring. (5 points)

3. a. Let p be a prime. Show that there exists a Galois extension K of the rationals $\mathbb{Q}$ with degree $(K : \mathbb{Q}) = p$. (5 points)

b. Let p_n denote the n^{th} prime. Prove that there exist fields F_i satisfying

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

and such that F_i is Galois over $\mathbb{Q}$ with $(F_i : F_{i-1}) = p_i$. (5 points)

4. Let K be a field.

a. Let S be a K-vector subspace of the space of $n \times n$ matrices $\mathrm{M}_n(K)$. Assume that every nonzero matrix in S is nonsingular. Prove that $\dim_K S \leq n$. (5 points)

b. Suppose there exists a field extension $E \supseteq K$ with degree $(E : K) = n$. Show that a subspace S of $\mathrm{M}_n(K)$ as above exists with $\dim_K S = n$. (5 points)

5. a. Let G be a finite simple group. If H is a subgroup of G with index 12 and if G contains an element of order 15, prove that H has an element of order 15. (5 points)

b. Show that any simple group of order 660 has exactly 66 Sylow-5 subgroups. (Hint. First show that the Sylow 11 normalizer has index 12.) (5 points)

6. Let R be a commutative ring with 1.

a. Suppose P_1 and P_2 are prime ideals of R and that $P_1 \cap P_2$ is a primary ideal. Show that either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. (3 points)

b. Let J be the Jacobson radical of R and let $0 \neq a \in R$. If the ideal aJ is primary, prove that its radical $\sqrt{aJ}$ contains J. (4 points)

c. If all ideals of R are primary, prove that R has at most one nonzero prime ideal. (3 points)

7. Let K be a field and let E be a splitting field over K for some polynomial $f(x) \in K[x]$ of degree n. Suppose the roots of f in E are $\alpha_1, \alpha_2, \ldots, \alpha_n$ where the α_i are distinct. If E is not generated over K by any $n-2$ of these roots, determine the Galois group $\mathrm{Gal}(E/K)$ (up to isomorphism) and prove that $f(x)$ is irreducible in $K[x]$.

8. Let V be a vector space over the field F with $\dim_F V = n < \infty$ and let $T\colon V \to V$ be a linear transformation. Let k be an integer with $1 \leq k < n$ and assume that $T(W) \subseteq W$ for all subspaces $W \subseteq V$ with $\dim_F W = k$. Prove that T is multiplication by some scalar $\alpha \in F$.