ALGEBRA QUALIFYING EXAM
January, 1989

Instructions: Do precisely 4 of the following 8 problems.

1. Let H be a finite group with trivial center. Show that the following are equivalent. (5 points for each implication)

i. Every automorphism of H is inner.

ii. Whenever H is a normal subgroup of a group G, then

$$G = H \times \mathbf{C}_G(H)$$

where $\mathbf{C}_G(H)$ is the centralizer of H in G.

2. Let R be a right Artinian ring with 1 and assume that every unit (that is, element of R with a two-sided inverse) is central. If $\mathrm{J}(R)$ is the Jacobson radical of R, prove that $R/\mathrm{J}(R)$ is commutative.

3. Let F be the finite field $\mathrm{GF}(7) = \mathbb{Z}/7\mathbb{Z}$. Prove that the polynomial $x^9 - 2$ is an irreducible factor of $x^{27} - 1$ in $F[x]$.

4. Let V be a finite dimensional vector space over some field F.

a. If $T: V \to V$ is an F-linear transformation such that $T^2 = T$, show that V is the direct sum $V = V_0 \dot{+} V_1$ where

$$T(v) = \begin{cases} 0 & \text{if } v \in V_0 \\ v & \text{if } v \in V_1. \end{cases}$$

(3 points)

b. Now assume F is a finite field $\mathrm{GF}(q)$ and that $\dim_F V = 3$. Determine (in terms of q) the number of linear transformations T with $T^2 = T$. (7 points)

5. Let G be a nonabelian finite simple group of order divisible by some prime p. If G has no more than $2p$ Sylow p-subgroups, determine (in terms of p) the number of elements of G whose order is a power of p.

6. Let M be a right R-module, where R is a ring with 1. Suppose $M = M_1 \dot{+} M_2 \dot{+} \cdots \dot{+} M_n$ is the internal direct sum of the simple submodules $M_1, M_2, \ldots, M_n$. Show that the following are equivalent.

i. The M_i are pairwise nonisomorphic.

ii. M has precisely n simple submodules.

iii. M has precisely 2^n submodules.

7. Let K be a finite degree extension field of the rationals $\mathbb{Q}$. Show that the following two statements are equivalent. (5 points for each implication)

i. K is a splitting field for some polynomial $g(x) \in \mathbb{Q}[x]$.

ii. For every irreducible polynomial $f(x) \in \mathbb{Q}[x]$, all irreducible factors of $f(x)$ in $K[x]$ have equal degrees.

8. Let $\mathbb{Q}^n$ denote the n-dimensional row vector space over the rationals $\mathbb{Q}$ and let $\mathbb{Z}^n$ denote the subset of those vectors with entries in the integers $\mathbb{Z}$. Notice that for each prime p there is a natural map $\theta_p: \mathbb{Z}^n \to (\mathbb{Z}/p\mathbb{Z})^n$. Show that for each $\mathbb{Q}$-linearly independent subset S of $\mathbb{Z}^n$ there are at most finitely many primes p such that the image $\theta_p(S)$ is not $\mathbb{Z}/p\mathbb{Z}$-linearly independent.