Algebra Qualifying Exam
September 1990

Do 4 of the following 8 problems.

1. Let H be a normal subgroup of the finite group G and assume that H is a nonabelian group of order pq where $p > q$ are primes. If G/H is a p-group, prove that G is the direct product of H with the centralizer of H in G.

2. Let F be a field, let $F[x]$ be the polynomial ring over F and let $F(x)$ be the field of rational functions in the variable x. Suppose $f \in F[x]$ is a nonconstant polynomial and, as usual, write $F[f]$ for the subring of $F[x]$ consisting of all polynomial expressions in f with coefficients in F. Furthermore, let $F(f) \subseteq F(x)$ be the field of fractions of $F[f]$.

a. (5 points) Show that $F[x]$ is an integral extension of $F[f]$.

b. (5 points) Prove that $F(f) \cap F[x] = F[f]$.

3. Let $F \subseteq \mathbb{C}$ be the splitting field for the polynomial $x^5 - 2$ over the rationals $\mathbb{Q}$.

a. (2 points) If $i = \sqrt{-1} \in \mathbb{C}$, show that $F[i]$ contains a primitive 20-th root of 1.

b. (5 points) Prove that $i \notin F$.

c. (3 points) Prove that the field $\mathbb{Q}[i + \sqrt[5]{2}]$ is not isomorphic to $\mathbb{Q}[\sqrt[5]{2}]$.

4. Let $f \in \mathbb{Q}[x]$ be a polynomial which is not solvable by radicals. Suppose that E is the splitting field of f over $\mathbb{Q}$ and that the degree $|E : \mathbb{Q}| = 168$.

a. (5 points) Show that the Galois group $G = \mathrm{Gal}(E/\mathbb{Q})$ has a subgroup of order 21.

b. (5 points) Prove that there exists a polynomial $g \in \mathbb{Q}[x]$ of degree 8 whose splitting field is also E.

5. Let $S = \mathrm{Sym}_n$ be the symmetric group on n letters and let C_1 and C_2 be cyclic subgroups of S of the same order m. Suppose $\theta: C_1 \to C_2$ is an isomorphism.

a. (5 points) If $C_1 = C_2$, so that θ is an automorphism of C_1, prove that there exists $s \in S$ such that

$$(*) \qquad \theta(c) = s^{-1}cs \qquad \text{for all } c \in C_1.$$

b. (5 points) Give an example to show that $(*)$ need not hold if $C_1 \neq C_2$. Hint. Cycle structures are relevant for both parts.

6. Let R be a right Artinian ring (with 1) and let J denote its Jacobson radical. Show that J is the set of all nilpotent elements of R if and only if R/J is a ring direct sum of division rings.

7. Suppose E is a field and $g(x) \in E[x]$ is a monic irreducible polynomial.

a. (4 points) Prove that each root of g has multiplicity which is equal to 1 or a power of the characteristic of E.

b. (3 points) Suppose F is a subfield of E and that some power of g is contained in $F[x]$. If $n \geq 1$ is minimal with $g^n \in F[x]$, prove that g^n is an irreducible polynomial of $F[x]$.

c. (3 points) If n is as in part (b), prove that n is either equal to 1 or to a power of the characteristic of E.

8. Let V be a vector space of finite dimension $n \geq 1$ over a field F of characteristic $\neq 2$ and let $\langle x, y \rangle$ be a nondegenerate symmetric bilinear form on V.

a. (4 points) Show that there exists $v \in V$ with $\langle v, v \rangle \neq 0$.

b. (4 points) Prove that V has a basis $\{ v_1, v_2, \ldots, v_n \}$ such that $\langle v_i, v_j \rangle = 0$ for $i \neq j$ and that $\langle v_i, v_i \rangle \neq 0$ for all i.

c. (2 points) If F is algebraically closed and $n \geq 2$, prove that there exists $0 \neq v \in V$ with $\langle v, v \rangle = 0$.