

**Algebra Qualifying Exam**  
**September 1991**

Do all 5 problems.

1. Let  $p$  and  $q$  be distinct primes and suppose  $G$  is a finite group having precisely  $p+1$  Sylow  $p$ -subgroups and  $q+1$  Sylow  $q$ -subgroups. Prove that there exist  $P \in \text{Syl}_p(G)$  and  $Q \in \text{Syl}_q(G)$  such that the subgroup of  $G$  generated by  $P$  and  $Q$  is  $PQ = P \times Q$ .

2. Let  $R$  be a commutative ring with 1. If  $a \in R$ , we write  $\text{ann}(a) = \{r \in R \mid ar = 0\}$  for the annihilator of  $a$  in  $R$ . Thus  $\text{ann}(a)$  is an ideal of  $R$  and we let  $S \subseteq R$  be the set of all elements  $a \in R$  such that  $\text{ann}(a)$  is a prime ideal of  $R$ .

- i. (4 points) If  $R$  is Noetherian, show that  $S$  is nonempty.
- ii. (4 points) If  $a \in S$  and  $r \in R$ , show that either  $ar = 0$  or  $ar \in S$ .
- iii. (2 points) If  $a, b \in S$  and  $\text{ann}(a) \neq \text{ann}(b)$ , prove that  $ab = 0$ .

3. Let  $F \subseteq E$  be an algebraic extension of fields. We say that an element  $\alpha$  of  $E$  is *abelian* if  $F[\alpha]$  is a Galois extension of  $F$  with abelian Galois group  $\text{Gal}(F[\alpha]/F)$ . Prove that the set of abelian elements of  $E$  is a subfield of  $E$  containing  $F$ .

4. Let  $V$  be a finite-dimensional vector space over a field  $K$  and let  $(\ , \ )$  be a bilinear form on  $V$ . Suppose  $T: V \rightarrow V$  is a linear transformation satisfying  $(v, Tw) = (Tv, w)$  for all  $v, w \in V$ . Write  $N = \ker(T) = \{v \in V \mid Tv = 0\}$ .

- i. (5 points) Assume that the form restricted to  $N$  is nondegenerate – that is, if  $v \in N$  and  $(v, N) = 0$ , then  $v = 0$ . If  $T$  is nilpotent, prove that  $T = 0$ .
- ii. (5 points) Find a 2-dimensional example with  $T$  a nonzero nilpotent transformation and with the form  $(\ , \ )$  nondegenerate on the whole vector space  $V$ .

5. Let  $R$  be a ring with 1 and let  $M$  be a right  $R$ -module. Suppose

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a chain of submodules such that, for  $i = 1, 2, \dots, n$ , the factors  $M_i/M_{i-1}$  are simple and pairwise nonisomorphic. If  $X$  and  $Y$  are isomorphic submodules of  $M$ , prove that  $X = Y$ .