

Algebra Qualifying Exam
September 1994

Do all 5 problems.

- 1. Let G be a finite group and let $P \in \text{Syl}_p(G)$ for some prime p . Suppose N is a normal subgroup of G with $|G : N| = |P| > 1$.
 - i. Prove that N is the subset of G consisting of all elements of order not divisible by p . (4 points)
 - ii. If the elements of G outside of N all have p -power order, prove that P is its own normalizer. (6 points)

2. Let R be a commutative ring and let P be a prime ideal of R . If V is a (right) R -module, define

$$W = \{v \in V \mid va = 0 \text{ for some } a \in R \setminus P\}$$

where $R \setminus P$ is the set of elements of R not in P .

- i. Show that W is an R -submodule of V . (2 points)
 - ii. If R is Noetherian and V is a finitely generated R -module, prove that $Wb = 0$ for some $b \in R \setminus P$. (3 points)
 - iii. If V is a simple R -module and $W = 0$, prove that P is a maximal ideal. (5 points)
3. Let E be a splitting field of the polynomial $x^3 - 2$ over the rationals \mathbb{Q} , and assume that E is contained in the complex numbers \mathbb{C} . Let $F = E \cap \mathbb{R}$ be the real subfield of E , and note that $F = \mathbb{Q}[\sqrt[3]{2}]$.
- i. Show that $G = \text{Gal}(E/\mathbb{Q})$ contains an element σ with the property that the only elements of F fixed by σ are rational. (4 points)
 - ii. Let $a \in F$ and suppose that $a^3 \in \mathbb{Q}$. Show that one of a , $a\sqrt[3]{2}$, or $a\sqrt[3]{4}$ is contained in \mathbb{Q} . (4 points)
 - iii. Prove that $\sqrt[3]{3} \notin E$. (2 points)

4. Let A be an $n \times n$ matrix over a field K and assume that the characteristic polynomial of A has distinct roots in the algebraic closure of K . Prove that any two $n \times n$ K -matrices which commute with A must commute with each other.

5. Let $S = M_n(F)$ be the ring of $n \times n$ matrices over the field F .

- i. If $s \in S$ is nilpotent, prove that the trace of s is zero. (4 points)
- ii. Suppose R is a ring and that $\theta: R \rightarrow S$ is a surjective ring homomorphism. Let I be an ideal of R with the property that every element of I is a sum of nilpotent elements of R . Show that $\theta(I) = 0$. (6 points)