Algebra Qualifying Exam
January 1995

Do all 5 problems.

1. Let P be a Sylow p-subgroup of a finite group G and let $N = \mathbb{N}_G(P)$ be its normalizer.
   i. Show that N is not contained in any proper normal subgroup of G. (5 points)
   ii. If the commutator subgroup G' is abelian, prove that $G' \cap N$ is normal in G. (5 points)

2. Let R be a commutative Noetherian domain and suppose that P is the unique nonzero prime ideal of R.
   i. Show that every element of R not in P is a unit of R. (3 points)
   ii. If Q is a nonzero ideal of R, prove that Q is primary and that $Q \supseteq P^n$ for some integer $n \geq 1$. (3 points)
   iii. If $P = (\pi)$ is principal, prove that every nonzero element of R is a product of a unit and a power of π. (4 points)

3. Let $\mathbb{Q}$ be the field of rational numbers and let $f(x) = x^8 + x^4 + 1$ be a polynomial in $\mathbb{Q}[x]$. Suppose E is a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \mathrm{Gal}(E/\mathbb{Q})$.
   i. Find $|E : \mathbb{Q}|$ and determine the Galois group G up to isomorphism. (5 points)
   ii. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of G on Ω. (5 points)

4. Let A be an $n \times n$ matrix over an algebraically closed field K and let $K[A]$ denote the K-linear span of the matrices $I = A^0, A, A^2, A^3, \ldots$. Show that A is diagonalizable if and only if $K[A]$ contains no nonzero nilpotent element.

5. Let G be a (not necessarily finite) group and denote the operation in G by multiplication. Let $\mathbb{Z}[G]$ denote the group ring of G over the integers $\mathbb{Z}$. Thus, every element of $\mathbb{Z}[G]$ is a finite $\mathbb{Z}$-linear combination of elements of G, and the multiplication in $\mathbb{Z}[G]$ is built naturally from the multiplication in G. Let I be a right ideal of $\mathbb{Z}[G]$ and define

$$\mathrm{gp}(I) = \{\, g \in G \mid 1 - g \in I \,\}.$$

   i. Prove that $\mathrm{gp}(I)$ is a subgroup of G. (4 points)
   ii. If I is a 2-sided ideal of $\mathbb{Z}[G]$, show that $\mathrm{gp}(I)$ is normal in G. (3 points)
   iii. If $\mathrm{gp}(I) = G$, prove that I is a 2-sided ideal of $\mathbb{Z}[G]$. (3 points)