

**Algebra Qualifying Exam
August 1996**

Do all 5 problems.

1. We say that a group G has property $(*)$ if every normal abelian subgroup of G is contained in $Z(G)$, the center of G .

- a. Suppose that N and M are normal subgroups of a group G and that G/N and G/M have property $(*)$. Prove that $G/(N \cap M)$ has property $(*)$. (3 points)
- b. Let $N \triangleleft G$ and assume that G/N has property $(*)$. If N has no nontrivial abelian normal subgroups, prove that G has property $(*)$. (3 points)
- c. Show that a finite p -group with property $(*)$ must be abelian. (4 points)

2. Let R be a commutative ring with 1, let $n \geq 2$ be a fixed integer, and suppose that $x^n = x$ for all $x \in R$.

- a. If P is a prime ideal of R , show that R/P is a finite field containing at most n elements. (4 points)
- b. Prove that the intersection of all prime ideals of R is the zero ideal. (2 points)
- c. If R is a Noetherian ring, conclude that R is finite. (4 points)

3. Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$, let α be a root of $f(x)$ in \mathbb{C} , and set $E = \mathbb{Q}[\alpha]$.

- a. Show that E contains a primitive 6th root of unity. (3 points)
- b. Prove that E is Galois over \mathbb{Q} . (2 points)
- c. Count the number of intermediate fields F with $\mathbb{Q} \subseteq F \subseteq E$ and $|F : \mathbb{Q}| = 3$. Justify your answer. (5 points)

4. Let V be a finite dimensional vector space over a field K and let $T : V \rightarrow V$ be a linear operator. Assume that there exists a nonzero vector $v \in V$ such that V is spanned by the vectors vT^i for $i = 0, 1, 2, \dots$

- a. Show that there exists a basis for V with respect to which the matrix of T has the form

$$\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & 1 & \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & \end{bmatrix}$$

for suitable $a_i \in K$. (5 points)

- b. Prove that the minimal polynomial and the characteristic polynomial of T are identical. (5 points)

5. The goal of this problem is to prove:

Theorem. Let $M_3(F)$ denote the space of 3×3 matrices over the field F . The following are equivalent.

- i. F has an extension field of degree 3.
 - ii. $M_3(F)$ contains a 3-dimensional subspace whose nonzero members are all invertible matrices.
 - iii. $M_3(F)$ contains a 2-dimensional subspace whose nonzero members are all invertible matrices.
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- a. If E is an extension field of F of degree 3, show that the ring $M_3(F)$ contains an isomorphic copy of E . Deduce that (i) implies (ii). (5 points)
 - b. Let A and B be linearly independent invertible matrices in $M_3(F)$. If the characteristic polynomial of AB^{-1} is not irreducible over F , show that some nonzero F -linear combination of A and B is not invertible. Deduce that (iii) implies (i). (5 points)