Algebra Qualifying Exam
January 1997

Do all 5 problems.

1. Let G be a finite group having the property that for every choice of two subgroups $X \subseteq G$ and $Y \subseteq G$, either $X \cap Y = 1$ or $X \subseteq Y$ or $Y \subseteq X$.
   i. If $H \subseteq G$, show that either $|H|$ is a prime power or else that $|H|$ and $|G : H|$ are relatively prime. (4 points)
   ii. If $1 < N \lhd G$, prove that G/N is nilpotent. (2 points)
   iii. If $N \lhd G$ and $N \neq G$, show that N is nilpotent. (4 points)

2. Let R be a ring, let V be a right R-module, and suppose that $V = V_1 \dotplus V_2 \dotplus V_3 \dotplus \cdots$ is the (internal) direct sum of its submodules $V_1, V_2, V_3, \ldots$. Show that V is an Artinian module if and only if each V_i is Artinian and only finitely many of the V_i's are nonzero.

3. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 5 over the rational numbers $\mathbb{Q}$ that is not solvable by radicals, and let S be the splitting field of $f(x)$ over $\mathbb{Q}$ which is contained in the complex numbers.
   i. Show that there exists at most one subfield E of S such that $|E : \mathbb{Q}| = 2$. (7 points)
   ii. If $\alpha, \beta \in S$ are irrational elements which satisfy $\alpha^2 \in \mathbb{Q}$ and $\beta^2 \in \mathbb{Q}$, prove that $\alpha\beta \in \mathbb{Q}$. (3 points)

4. If K is a field, then the general linear group $G = \mathrm{GL}_n(K)$ is the multiplicative group of $n \times n$ invertible matrices over K.
   i. If the characteristic of K is not equal to 2, show that G has precisely n conjugacy classes of elements of order 2. (5 points)
   ii. If char $K = 2$, show that G has precisely $[n/2]$ (the greatest integer in $n/2$) conjugacy classes of elements of order 2. (5 points)

5. Let S be a commutative integral domain and let R be a subring of S with the same identity 1. Suppose that there exist finitely many elements $s_1, s_2, \ldots, s_n \in S$ such that $S = s_1R + s_2R + \cdots + s_nR$. Show that R is a field if and only if S is a field.