

**Algebra Qualifying Exam**  
**January 1998**

Do all 5 problems.

1. Fix a prime  $p$  and let  $G$  be a finite group with the property that every nonidentity  $p$ -subgroup of  $G$  is contained in a unique Sylow  $p$ -subgroup of  $G$ . Suppose  $N \triangleleft G$  and  $|N|$  is divisible by  $p$ .
  - i. If  $P$  and  $Q$  are Sylow  $p$ -subgroups of  $G$ , show that  $Q = P^n$  for some element  $n \in N$ . (6 points)
  - ii. Prove that  $G/N$  has a unique Sylow  $p$ -subgroup. (4 points)
  
2. Let  $R$  be a commutative domain and write  $(a)$  for the principal ideal generated by  $a \in R$ . Recall that an element of  $R$  is said to be irreducible if it is nonzero, not a unit, and has no proper factorization.
  - i. Show that  $(a) \subseteq (b)$  if and only if  $b \mid a$ , and that  $(a) = (b)$  if and only if  $b = au$  for some unit  $u \in R$ . (2 points)
  - ii. If  $R$  is a UFD (unique factorization domain), prove that the set of principal ideals of  $R$  satisfies the maximal condition. (4 points)
  - iii. If the set of principal ideals of  $R$  satisfies the maximal condition, show that every nonzero, nonunit element of  $R$  can be written as a finite product of irreducible elements. (4 points)
  
3. Let  $p$  be a prime, let  $F \subseteq K$  be fields of characteristic 0, and assume that  $F$  contains a primitive  $p$ th root of unity. Fix  $a \in K$ .
  - i. Prove that there exists a field  $E \supseteq K$  such that  $E$  contains a  $p$ th root of  $a$  and  $|E : K| = 1$  or  $p$ . (4 points)
  - ii. Now assume that  $K$  is a finite degree Galois extension of  $F$ . Show that there exists a field  $E \supseteq K$  such that  $E$  contains a  $p$ th root of  $a$ ,  $E$  is Galois over  $F$ , and  $|E : K|$  is a power of  $p$ . (6 points)
  
4. Let  $V$  be a finite dimensional vector space over a field of characteristic 0. Suppose  $T: V \rightarrow V$  is a linear operator such that the trace  $\text{tr } T^k = 0$  for all integers  $k \geq 1$ .
  - i. Show that the constant term of the characteristic polynomial of  $T$  is zero, and deduce that  $T(V) \neq V$ . (5 points)
  - ii. Let  $S$  denote the restriction of  $T$  to the subspace  $T(V)$ , so that  $S$  is a linear operator on  $T(V)$ . Prove that  $\text{tr } S^k = 0$  for all integers  $k \geq 1$ . (4 points)
  - iii. Show that  $T$  is nilpotent. (1 point)
  
5. Let  $G$  be a (not necessarily finite) group and let  $\theta: G \rightarrow G$  be a homomorphism such that  $\theta^n(G) = \{1\}$  for some integer  $n \geq 1$ .
  - i. If the kernel of  $\theta$  is finite, prove that the kernel of  $\theta^2$  is finite, and deduce that  $G$  is finite. (5 points)
  - ii. If  $\theta(G)$  has finite index in  $G$ , prove that  $\theta^2(G)$  has finite index in  $G$ , and deduce that  $G$  is finite. (5 points)