QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday, January 16, 2002

Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

1. (i) Let $\{f_n\}$ be a sequence of C^1 functions on a compact interval I such that $|f_n(x)| + |f_n'(x)| \leq M$ for all $x \in I$ and $n = 1, 2, 3, \ldots$. Show that there is a subsequence $\{f_{n_k}\}$ which converges uniformly on I.

(ii) Is the preceding statement still true if we drop the assumption that I is compact? (Proof or counterexample)

(iii) Can one also show that under the assumptions in (i) the sequence f_n has a subsequence whose derivatives converge uniformly? (Proof or counterexample)

2. Let $\{a_n\}_{n=1}^{\infty}$ be a numerical sequence and let

$$b_n = \frac{1}{n^6} \sum_{k=1}^{n} k^5 a_k.$$

(i) Prove or disprove: If a_n converges then b_n converges.

(ii) Prove or disprove: If b_n converges then a_n converges.

Hint: Relate $\sum_{k=1}^{n} k^5$ to an integral.

3. (i) Suppose that $\mathcal{O} \subset R^n$ is open, $f : \mathcal{O} \to \mathbb{R}$ is a C^∞ function and $n > 1$. Show that f is not a one to one function.

(ii) Suppose that $\mathcal{O} \subset \mathbb{R}^n$ is open, $f : \mathcal{O} \to \mathbb{R}^k$ is a C^∞ function and $n > k$. Show that f is not a one to one function.

Hint: Use induction on k.

4. Suppose that the sequence $\{f_n\}$ of nonnegative Lebesgue measurable functions on $\mathbb{R}$ converges to f pointwise, and suppose that $\int_{\mathbb{R}} f_n(x)dx < \infty$ for all n, $\int_{\mathbb{R}} f(x)dx < \infty$ and $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} f(x)dx$.

Prove that for all measurable sets E we have

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx$$

5. (i) Let $f : \mathbb{R} \to \mathbb{C}$ be continuous on $[0, 1]$ and assume that f is 1-periodic, *i.e.* $f(x + 1) = f(x)$ for all x. Let β be an *irrational* number in $(0, 1)$ and define

$$\mathfrak{S}_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(x + k\beta)$$

Show that for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathfrak{S}_n f(x) = \int_0^1 f(t)dt$$

and that the convergence is uniform on $\mathbb{R}$.

Hint: Show first that the formula for the limit is correct for $f_m(x) = e^{2\pi i m x}$, $m \in \mathbb{Z}$.

(ii) Formulate and prove a generalization for $f \in L^p$, for $1 \leq p < \infty$.

6. (i) Show that for nonnegative scalars $a, b \in \mathbb{R}$ and $p \geq 2$ we have

$$a^p + b^p \leq (a^2 + b^2)^{p/2}$$

and

$$\left(\frac{a^2 + b^2}{2}\right)^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.$$

Hint: For the second inequality use the convexity of $t \mapsto t^{p/2}$ for $t > 0$.

(ii) Show that for $f, g \in L^p(X, d\mu)$ and $2 \leq p < \infty$

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{2}\right\|_p^p \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.$$

(iii) Show that each closed convex set in L^p $(2 \leq p < \infty)$ has an element f of minimal norm.

7. Suppose u is real-valued and harmonic in $\mathbb{C}$ and suppose that for all $r > 0$

$$\max_{0 \leq \theta \leq 2\pi} u(re^{i\theta}) \leq M(r),$$

with $0 \leq M(r) < \infty$.

a) Show that

$$\frac{1}{2\pi}\int_0^{2\pi} |u(re^{i\theta})| d\theta + u(0) \leq 2M(r).$$

b) Suppose that f is an entire function and there are non-negative constants A, B and λ so that

$$Re\ f(re^{i\theta}) \leq Ar^\lambda + B$$

for all r and θ. Show that f is a polynomial of degree at most λ.

8. Evaluate

$$\int_0^\infty \frac{(\ln x)^2}{1 + x^2} dx.$$

9. Fix $0 < r < R$ and let $A = \{z \in \mathbb{C} : r < |z| < R\}$. Suppose that f is holomorphic and non vanishing on A and continuous on the closure of A and that $|f(re^{i\theta})| \equiv \alpha$ and $|f(Re^{i\theta})| \equiv \beta$ for some constants α and β and for all $0 \leq \theta \leq 2\pi$. Show that $f(z) = cz^n$ for some $c \in \mathbb{C}$ and some $n = 0, \pm 1, \pm 2, \cdots$.

Hint: Consider the function $u(z) = log|f(z)| - \gamma log|z|$ for an appropriate value of γ.