QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday January 15, 2003

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.

(2) $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ denotes the unit disc in the complex plane.

(3) For points x and y in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.

(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $\big|E\big|$ denotes its Lebesgue measure.

(5) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then for $1 \leq p < +\infty$,

$$||f||_p = \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p}.$$

Two functions on X are said to be equivalent if they are equal except on a set of μ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(6) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then

$$||f||_\infty = \inf\left\{t > 0 \,\big|\, \mu(\{x \in X \,\big|\, |f(x)| > t\}) = 0\right\}.$$

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on X such that $||f||_\infty < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset\subset \mathbb{R}$.

(8) If f and g are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x - t)\, g(t)\, dt$$

whenever the integral converges.

(9) If T is a distribution and φ is a test function, then $\langle T, \varphi\rangle$ denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

Problem I Does the series

$$\sum_{k=1}^{\infty} \frac{\sin \sqrt{k}}{k}$$

converge?

[*Hint: compare* $\sum_{k=M}^{N} \frac{\sin\sqrt{k}}{k}$ *with a similar integral.*]

Problem II Let $E \subset \mathbb{Q}$ be the set of x whose decimal expansion is of the form $x = 0.d_1 d_2 \cdots d_N$ for some $N \in \mathbb{N}$, and where $d_1, \ldots, d_N \in \{1,2,3,4,5,6,7,8\}$ (so $d_i \neq 0$ and $d_i \neq 9$ for $i = 1, \ldots, N$). Show that any compact subset of E is finite.
Can we drop the hypotheses that $d_i \neq 0$ and $d_i \neq 9$?

Problem III Let $Q = [0,1] \times \cdots \times [0,1] \subset \mathbb{R}^n$ be the unit cube, and consider the function

$$f(x_1, \ldots, x_n) = \frac{x_1 x_2 \cdots x_n}{x_1^{a_1} + \cdots + x_n^{a_n}},$$

where the a_j are positive constants. For which $a_1 > 0$, ..., $a_n > 0$ is the integral $\int_Q f(x)dx$ finite?

Problem IV Let $1 \leq p < \infty$, and let $f_n \in L^p(\mathbb{R})$ be a sequence of functions. Suppose

$$\sum_{n=1}^{\infty} \|f_{n+1} - f_n\|_{L^p} < \infty.$$

Show that the sequence f_n converges pointwise almost everywhere.

Problem V Define

$$\Lambda(x) = \int_0^{\infty} \frac{e^{-t}}{\log t}\left(t^{x-1} - 1\right)dt.$$

(1) For which $x \in \mathbb{R}$ is the integrand a Lebesgue integrable function?
(2) Show that $\Lambda(x)$ is a continuous function for $x \in \mathbb{R}_+$.
(3) Show that $\Lambda(x)$ is differentiable for $x \in \mathbb{R}_+$, and that the derivative is given by $\Lambda'(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$.

Give complete proofs.

Problem VI

Let $K : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative measurable function for which

$$\int_0^\infty \frac{K(t)}{\sqrt{t}} dt = A < \infty.$$

In this problem $L^2(0,\infty) = \{f : (0,\infty) \to \mathbb{R} : f \text{ is measurable and } \int_0^\infty f(x)^2 dx < \infty\}$, with the usual convention that functions which differ only on a set of measure zero are identified.

(1) Show that for any two functions $f, g \in L^2(0,\infty)$ one has

$$\int_0^\infty \int_0^\infty K(xy) f(x) g(y)\, dx\, dy \le A \, \|f\|_{L^2} \, \|g\|_{L^2}.$$

[*Hint: try the substitution $x = z/y$; or you could try to solve (b) first...*]

(2) Prove that for any $f \in L^2(0,\infty)$ the integral

$$Tf(x) \stackrel{\text{def}}{=} \int_0^\infty K(xy) f(y) dy$$

converges for almost every $x \in \mathbb{R}_+$, and that T defines a bounded operator on $L^2(0,\infty)$ (i.e. there is a finite constant C such that $\|Tf\|_{L^2} \le C\|f\|_{L^2}$ for all $f \in L^2(0,\infty)$.)

Problem VII

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.

(1) Let (h_n) be a sequence of holomorphic maps from $\mathbb{D}$ into $\mathbb{D}$. Assume that $|h_n'(0)|$ tends to 1 as n tends to ∞. Show that $h_n(0)$ tends to 0.

(2) Let $\mathcal{F}$ be the set of holomorphic maps from $\mathbb{D}$ into $(\mathbb{D} - \{\frac{1}{2}\})$. Show that there is a constant $M < 1$ such that for every $f \in \mathcal{F}$, $|f'(0)| \le M$.

The second question can be treated, assuming the result of the first question.

Problem VIII

Let $0 < \epsilon < \infty$, $0 < R < \infty$ and let D_1, D_2 be two closed disjoint disks in $\mathbf{C}$. Show that there is an entire function f so that $f(D_1)$ is contained in $\{z : |z| < \epsilon\}$ and $f(D_2)$ contains $\{z : |z| < R\}$.

[*Hint: Use Runge's theorem.*]

Problem IX

Use complex methods to find the value of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$