QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison

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Version for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.

(2) $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ denotes the unit disc in the complex plane.

(3) For points x and y in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.

(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.

(5) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then for $1 \leq p < +\infty$,

$$||f||_p = \Big[\int_X |f(x)|^p \, d\mu(x) \Big]^{1/p}.$$

Two functions on X are said to be equivalent if they are equal except on a set of μ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(6) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then

$$||f||_\infty = \inf \{ t > 0 \,|\, \mu(\{x \in X \,|\, |f(x)| > t\}) = 0 \}.$$

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on X such that $||f||_\infty < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset\subset \mathbb{R}$.

(8) If f and g are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x - t) \, g(t) \, dt$$

whenever the integral converges.

(9) If T is a distribution and φ is a test function, then $\langle T, \varphi \rangle$ denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

ADVANCED CALCULUS

Problem I

Let $U \subset \mathbb{R}^3$ be a nonempty open subset.

Use differential calculus to show that a continuously differentiable map $f : U \to \mathbb{R}^2$ cannot be injective.

Problem II Let $\varepsilon_n > 0$ be a sequence with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

(a) Suppose that u_n is a sequence of real numbers satisfying

$$u_{n+1} \le u_n + \varepsilon_n$$

for all $n \ge 1$. Show that $\lim_{n\to\infty} u_n$ exists (the possibility $\lim_{n\to\infty} u_n = -\infty$ is allowed.)

(b) Suppose that $v_n > 0$ are real numbers satisfying $v_n \le 1 + \varepsilon_n$. Show that $\lim_{n\to\infty} \prod_{k=1}^{n} v_k$ exists.

Problem III According to a Theorem of Weierstrass, every continuous function on $[-1, +1]$ can be uniformly approximated by a sequence of polynomials. Here we study the question of approximation by polynomials of fixed degree.

Let f be a C^4 function defined on $[-1, +1]$ (i.e. f and its derivatives of order ≤ 4 are continuous functions on $[0, 1]$.) Show that there is a constant $C > 0$ such that for every polynomial P of degree ≤ 4

$$\sup_{|x|\le 1} |f(x) - P(x)| \ge C \left| \int_{-1}^{+1} x(x^2-1)^4 \, f''''(x) \, dx \right| .$$

Either give an explicit value of C or indicate very clearly an easy computation that would lead to such a value. Give full justifications.

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Problem IV Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which vanishes outside the unit circle and define

$$I(f) = \int_0^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta) \, dr \, d\theta.$$

For which $p \ge 1$ is there a constant C_p such that

$$I(f) \le C_p \|f\|_{L^p(\mathbb{R}^2)}$$

for all such functions f.

Problem V Let $f \in L^\infty(\mathbb{R})$ with $f(x+1) = f(x)$.

(a) Show that for every measurable subset $E \subset [0, 1]$ we have

$$\lim_{n\to\infty} \int_E f(nx)dx = |E| \int_0^1 f(x)dx.$$

Hint: one approach is to first show that it is true for the functions $f_k(x) = e^{2\pi i k x}$, $k = 0, \pm 1, \pm 2 \cdots$.

(b) Suppose that there is a measurable set $E \subset [0, 1]$ with $|E| > 0$ such that for some sequence of integers $n_k \to \infty$,

$$\lim_{k\to\infty} f(n_k x) = g(x)$$

exists for all $x \in E$. Show that there is a constant C such that $f(x) = C$ almost everywhere on $[0, 1]$.

Hint: First use part (a) to show that there is a constant C such that $g(x) = C$ almost everywhere on E. Then use part (a) again (with a different f) to show that $f(x) = C$ almost everywhere on $[0, 1]$.

Problem VI

(a) *True or false?* In other words, prove or disprove the following statements:

(*i*) If μ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \nearrow x_0} \mu((-\infty, x]) = \mu((-\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

(*ii*) If μ is a finite Borel measure on $\mathbb{R}$, then $\lim_{x \searrow x_0} \mu((-\infty, x]) = \mu((-\infty, x_0])$ holds for any $x_0 \in \mathbb{R}$.

Let $E \subset \mathbb{R}$. Let f be a continous function on $\mathbb{R}$. Assume that f is differentiable at any point $x \in \mathbb{R} \setminus E$, and that for any such point $f'(x) = 0$.

(b) Assume that the set E has Lebesgue measure 0, must f be constant?

(c) Assume that the set E is countable. Show that f is constant. Although the result is true in this generality, full credit will be given for a proof in the special and easier case when E is a *closed* countable set.

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Problem VII Let $f_n(z)$ be a sequence of polynomials. Assume that for some function $h : \mathbb{C} \to \mathbb{C}$ one knows that

$$\lim_{n\to\infty} f_n^2(z) + f_n(z) = h(z)$$

uniformly on each compact subset of $\mathbb{C}$.

(a) Show that $h(z)$ is not the polynomial $h(z) = z$.

(b) If $h(z) = az^2 + bz + c$, find all possible values of a, b, c (or a necessary and sufficient condition on a, b, c).

Problem VIII Suppose that f is holomorphic in the unit disc in C and that

$$\int_0^{2\pi} |f(re^{it})|^p dt \le \frac{C}{(1-r)^A}$$

for some $1 < p < \infty$ and some constants $C > 0$ and $A \ge 0$. Show that $|f(z)| \le \dfrac{D}{(1-|z|)^B}$ for some positive constants D and B. Try to find the best value of B.

Problem IX Let g be a continuous function defined on the interval $[-1, +1]$ in $\mathbf{R}$. It is a classical result that if one sets

$$g_\tau(x) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi\tau}} g(t) e^{\frac{-(x-t)^2}{2\tau}} \, dt \; ,$$

then for any $x \in (-1, +1)$, $g_\tau(x)$ tends to $g(x)$ as $\tau \to 0^+$, and the convergence is uniform on smaller intervals.

(It is just a matter of classical approximate identity kernels, it shows up naturally when solving the heat equation and it was used by Weierstrass in proving his approximation theorem.)

Now, let g be a holomorphic function defined on $\mathbb{C}$. For $z \in \mathbb{C}$ and $\tau > 0$ set:

$$g_\tau(z) = \int_{-1}^{+1} \frac{1}{\sqrt{2\pi\tau}} g(t) e^{\frac{-(z-t)^2}{2\tau}} \, dt \; .$$

(a) Prove that g_τ is an entire function, i.e. a holomorphic function defined on all of $\mathbb{C}$.

(b) Find a region $U \subset \mathbb{C}$ containing a neighborhood of 0 such that for all $z \in U$, $g_\tau(z)$ tends to $g(z)$ as $\tau \to 0^+$.

Hint: If $z = x + iy$, switch from integration on [-1,+1] to integration on the line segment $[-1 + iy, 1 + iy]$.