QUALIFYING EXAM

in

ANALYSIS

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Tuesday January 10, 2006

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.

(2) $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ denotes the unit disc in the complex plane.

(3) For points x and y in $\mathbb{R}^n$, $|x-y|$ denotes the Euclidean distance between the points.

(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $\big|E\big|$ denotes its Lebesgue measure.

(5) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then for $1 \leq p < +\infty$,

$$||f||_p = \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p}.$$

Two functions on X are said to be equivalent if they are equal except on a set of μ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(6) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then

$$||f||_\infty = \inf\big\{t > 0 \,\big|\, \mu\big(\{x \in X \,\big|\, |f(x)| > t\}\big) = 0\big\}.$$

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on X such that $||f||_\infty < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset\subset \mathbb{R}$.

(8) If f and g are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x-t)\, g(t)\, dt$$

whenever the integral converges.

(9) If T is a distribution and φ is a test function, then $\langle T, \varphi\rangle$ denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

Problem I

(a) Determine all values of α, β such that the (possibly) improper integral

$$\int_0^1 x^\alpha \sin(x^\beta)\, dx$$

converges.

(b) Determine all values of α, β such that the improper integral

$$\int_0^1 x^\alpha |\sin(x^\beta)|\, dx$$

converges.

Problem II

(a) Suppose $f \in C^2((-\epsilon, \epsilon))$. Define

$$F(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y}, & x \neq y, \\ f'(x), & x = y. \end{cases}$$

Show that $F \in C^1((-\epsilon,\epsilon) \times (-\epsilon,\epsilon))$.

(b) Suppose $f \in C^2((-\epsilon,\epsilon))$ and $f(0) = f'(0) = 0$ and $f''(0) = 1$. Show that there exist $\delta, \eta > 0$ and $\varphi \in C^1((-\delta,\delta))$ such that $|\varphi(x)| < \eta$ for $|x| < \delta$ and $\varphi'(0) = -1$ and $f(\varphi(x)) = f(x)$.

(c) Show that the above φ satisfies $\varphi(\varphi(x)) = x$ for $|x| < \eta'$ and for some $\eta' \in (0, \eta]$.

Problem III Let a_k be a sequence of non-negative numbers satisfying $0 < \sum_{k=1}^{\infty} a_k < \infty$. Show that

$$\lim_{x \to +\infty} \sum_{k=1}^{\infty} a_k \sin \frac{x}{k}$$

does not exist.

Problem IV Show that if $f \in L^1(\mathbb{R})$ then

$$\lim_{\lambda \in \mathbb{R},\ \lambda \to \infty} \int_{\mathbb{R}} f(x) e^{-i\lambda x}\, dx = 0.$$

Problem V Let Ω be an open subset of $\mathbf{R}^n$. Let $1 \leq p \leq q \leq r \leq \infty$, and let $L^p = L^p(\Omega)$, $L^q = L^q(\Omega)$, and $L^r = L^r(\Omega)$.

(a) Show that

$$L^p \cap L^r \subset L^q \subset L^p + L^r.$$

By definition, $L^p + L^r = \{g + h : g \in L^p, h \in L^r\}$.

(b) Define canonical norms on $L^p \cap L^r$, $L^p + L^r$ by

$$\|f\|_{L^p \cap L^r} = \max\{\|f\|_{L^p}, \|f\|_{L^r}\},$$

$$\|f\|_{L^p + L^r} = \inf\{\|g\|_{L^p} + \|h\|_{L^r} : f = g + h, g \in L^p, h \in L^r\}.$$

(You don't need to verify the above two are norms.)

Prove that two inclusions in (a) are continuous maps.

Problem VI Assume $\{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+)$, i.e. $a_1^2 + a_2^2 + \cdots < \infty$ and $b_1^2 + b_2^2 + \cdots < \infty$.

(a) Using the Cauchy-Schwarz inequality show that

$$\sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_n b_m}{m}$$

is convergent.

(b) Show that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m}$$

is convergent.

(c) If instead of assuming $\{a_n\}, \{b_n\} \in l^2(\mathbb{Z}_+)$ we assume $a_n \in l^p(\mathbb{Z}_+)$ and $b_n \in l^{p'}(\mathbb{Z}_+)$, with $p, p' \in (1, \infty)$, $1/p + 1/p' = 1$, prove that (b) still holds.

Problem VII Suppose that f is holomorphic in $\{z = x + iy : y > 0\}$ and that $\lim_{z \to 0} f(z) = L$ exists (and is finite). Let $S = \{z = x + iy \colon y > |x|\}$. Show that

$$\lim_{S \ni z \to 0} z f'(z) = 0.$$

Problem VIII

(a) Show that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

(b) Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Problem IX Show that there is no entire function $f(z)$ satisfying

$$| f(z) - e^{\bar{z}} | \leq 3|z|, \quad z \in \mathbf{C}.$$