QUALIFYING EXAM

in

ANALYSIS

Department of Mathematics
University of Wisconsin-Madison
Wednesday, August 23, 2006

Versions for Math 722

Instructions: Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

Standard notation used on the Analysis exams:

(1) $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers respectively.

(2) $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ denotes the unit disc in the complex plane.

(3) For points x and y in $\mathbb{R}^n$, $|x - y|$ denotes the Euclidean distance between the points.

(4) If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set, then $|E|$ denotes its Lebesgue measure.

(5) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then for $1 \leq p < +\infty$,

$$||f||_p = \left[\int_X |f(x)|^p \, d\mu(x)\right]^{1/p}.$$

Two functions on X are said to be equivalent if they are equal except on a set of μ measure zero. For $1 \leq p < +\infty$, $L^p(X) = L^p(X, d\mu)$ is the space of equivalence classes of complex valued measurable functions such that $||f||_p < +\infty$.

(6) If μ is a positive measure on a set X, and f is a complex valued measurable function on X, then

$$||f||_\infty = \inf\left\{t > 0 \,\big|\, \mu(\{x \in X \,\big|\, |f(x)| > t\}) = 0\right\}.$$

$L^\infty(X)$ is the space of equivalence classes of measurable, complex valued functions on X such that $||f||_\infty < +\infty$.

(7) $L^p_{\text{loc}}(\mathbb{R})$ is the space of measurable, complex valued functions on $\mathbb{R}$ which belong to $L^p(K)$ for every compact set $K \subset\subset \mathbb{R}$.

(8) If f and g are measurable functions on $\mathbb{R}$, the convolution $f * g$ is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x - t)\, g(t)\, dt$$

whenever the integral converges.

(9) If T is a distribution and φ is a test function, then $\langle T, \varphi\rangle$ denotes the value of the distribution applied to the test function.

The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.

Problem I Give an example of a Riemann integrable function $f\colon [0,1] \to [0,1]$ which has a dense set of discontinuities. Verify all conclusions.

Problem II Let

$$f(x,y) = \sum_{n=1}^{\infty} \frac{x}{x^2 + yn^2}, \quad y > 0.$$

(a) Show that for each $y > 0$, $g(y) = \lim_{x \to +\infty} f(x,y)$ exists. Evaluate the limit function $g(y)$.

(b) Determine if $f(x,y)$ converges to $g(y)$ uniformly for $y \in (0,\infty)$ as $x \to +\infty$.

(Justify all steps.)

Problem III Let

$$s_n(x) = \sum_{k=1}^{n} \sin(kx).$$

Show that there exists a constant C, independent of N, x, such that

$$\sum_{n=1}^{N} \frac{|s_n(x)|}{n^2} < C, \quad 0 < x < \pi, \quad N = 1,2,3,\ldots.$$

(*Hint:* Estimate $s_n(x)$ for $n \le \frac{1}{x}$ and for $n > \frac{1}{x}$ separately.)

Problem IV Give an example of a sequence f_k such that f_k converges weakly to zero in $L^2[0,1]$ and strongly to zero in $L^{3/2}[0,1]$, but does not converge strongly in $L^2[0,1]$. Verify all conclusions.

Problem V Fix a function $g \in L^1(\mathbf{R})$ such that $\int g(x)\,dx = 0$. Denote $g_\epsilon(x) = \epsilon^{-1} g(\epsilon^{-1} x)$. Consider an operator

$$T_\epsilon f(x) = \int_{\mathbf{R}} g_\epsilon(y) f(x-y)\,dy.$$

(a) Prove that there exists a constant C such that $\|T_\epsilon f\|_p \le C\|f\|_p$, for all $\epsilon \ne 0$ and $1 \le p < \infty$.

(b) Prove that $\lim_{\epsilon \to 0} \|T_\epsilon f\|_p = 0$ for any $f \in L^p(\mathbf{R})$ with $1 \le p < \infty$.

Problem VI Fix $\alpha > 0$.

(a) Suppose that $f_n \in L^\infty(\mathbf{R})$ satisfy $\|f_n\|_{L^\infty(\mathbf{R})} \ge n^{1+\alpha}$, $n = 1,2,\ldots$. Show that there is a function $g \in L^1(\mathbf{R})$ such that $\lim_{n\to\infty} \|f_n g\|_{L^1(\mathbf{R})} = \infty$.

(b) Prove or disprove that (a) holds when $\alpha = 0$.

Problem VII Let $f(z) = 10z + z^2 + iz^4$.

(a) Show that for each w with $|w| < 8$, $f(z) = w$ has a unique solution z satisfying $|z| < 1$.

(b) Show that there exist distinct z_1, z_2 in $\{z\colon |z| < 2\}$ such that $f(z_1) = f(z_2)$.

Problem VIII

(a) Let $f(z)$ be the branch of $\sqrt{z(1-z)}$ on $\mathbf{C} \setminus [0,1]$ with $f(2) = \sqrt{2}i$. Determine the values of

$$\lim_{y \to 0, y>0} f(\frac{1}{2} + iy), \quad \lim_{y \to 0, y<0} f(\frac{1}{2} + iy).$$

(b) Evaluate

$$\int_0^1 \frac{x^2}{\sqrt{x(1-x)}}\, dx.$$

(Justify all steps.)

Problem IX Let $\Delta = \{z\colon |z| < 1\}$.

(a) Let $0 < a_n < 1$ such that $\sum_{n=1}^{\infty}(1 - a_n)$ is convergent. Show that the limit function

$$f(z) = \lim_{n \to +\infty} \prod_{k=1}^{n} \frac{a_k - z}{1 - a_k z}, \quad z \in \Delta$$

is holomorphic, and that f has zeros at a_n only.

(b) Give an example of a bounded holomorphic function g on Δ and a sequence b_n in Δ such that g has simple zeros at $b_n, n = 1, 2, \ldots$ and

$$\lim_{n \to \infty} g'(b_n)(1 - |b_n|) = 0.$$

Verify all conclusions.