Qualifying Exam in Analysis
Real and Complex Analysis (Math 721-722) Version
Wednesday, January 17, 2007

Instructions: *Do six of the nine problems. To receive credit on a problem. you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).*

1. Let X be a metric space with metric d.
(i) Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Prove that ρ is a metric on X.
(ii) Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ.

2. Let $u : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth function and let $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ be the Laplacian of u.
Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x,y,z) = x^3 y^3$ on the sphere of radius R centered at the origin. Find $u(0,0,0)$.

3. Let I be a compact subset of $(0, 2\pi)$. Show that the series

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

converges uniformly on I.

4. Let F be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of x to F, i.e. $\delta_F(x) = \inf\{|x-y| : y \in F\}$.

(i) Prove that δ_F is Lipschitz continuous, in fact

$$|\delta_F(x) - \delta_F(y)| \le |x-y|.$$

(ii) Let

$$M(x) = \int \frac{\delta_F(y)}{|x-y|^2} dy.$$

Show that $M(x) < \infty$ for almost every $x \in F$.
Hint: For part (ii) consider the integral $\int_F M(x)dx$.

5. On the interval $[-1,1]$ consider the standard Banach spaces L^1 and L^2 with the norms

$$\|f\|_{L^1} = \int_{-1}^{1} |f(x)|\, dx; \qquad \|f\|_{L^2} = \Big(\int_{-1}^{1} |f(x)|^2\, dx \Big)^{1/2}.$$

Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in L^2. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and

$$\big| \|f_j\|_{L^2} - \sqrt{2} \big| \leq 2^{-j}.$$

Show that $\lim_{j\to\infty} f_j(x) = 1$ for almost every $x \in [-1,1]$.

Hint: Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that f_n converges weakly to $f \in L^2$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx \quad \textit{for all } g \in L^2(\mathbb{R}).$$

Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. For $z \in \mathbb{C}$ evaluate

$$\frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - z|\, d\theta.$$

Suggestion: Treat the easier case $|z| > 1$ first.

8. Let $S = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, -1 < y < 1\}$, and let $f : S \to \mathbb{C}$ be a holomorphic function which satisfies the inequality

$$|f(z)| \leq 1 + |z|^2 \quad \text{for all } z \in S.$$

Show that for any $n = 0, 1, \ldots$ there is a constant C_n such that

$$|f^{(n)}(x)| \leq C_n(1 + |x|^2) \quad \text{for all } x \in \mathbb{R}.$$

What can you say about the constant C_n?

9. Let E denote a compact subset of $\mathbb{R}$ of measure 0 (here measure refers to Lebesgue measure on the real line). Let $f : \mathbb{C} \setminus E \to \mathbb{C}$ be a holomorphic function. Show that if f is bounded on any bounded subset of $\mathbb{C} \setminus E$, then f extends to a holomorphic function on $\mathbb{C}$.