

Qualifying Exam in Analysis
Real Analysis (Math 721-725) Version
Wednesday, January 17, 2007

Instructions: Do six of the nine problems. To receive credit on a problem, you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).

1. Let X be a metric space with metric d .

(i) Define $\rho : X \times X \rightarrow \mathbb{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that ρ is a metric on X .

(ii) Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ .

2. Let $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ denote a smooth function and let $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ be the Laplacian of u .

Suppose that $\Delta u = 1$ on \mathbb{R}^3 and $u(x, y, z) = x^3 y^3$ on the sphere of radius R centered at the origin. Find $u(0, 0, 0)$.

3. Let I be a compact subset of $(0, 2\pi)$. Show that the series

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

converges uniformly on I .

4. Let F be a closed set in \mathbb{R} whose complement has finite measure, and let $\delta_F(x)$ denote the distance of x to F , i.e. $\delta_F(x) = \inf\{|x - y| : y \in F\}$.

(i) Prove that δ_F is Lipschitz continuous, in fact

$$|\delta_F(x) - \delta_F(y)| \leq |x - y|.$$

(ii) Let

$$M(x) = \int \frac{\delta_F(y)}{|x - y|^2} dy.$$

Show that $M(x) < \infty$ for almost every $x \in F$.

Hint: For part (ii) consider the integral $\int_F M(x) dx$.

5. On the interval $[-1, 1]$ consider the standard Banach spaces L^1 and L^2 with the norms

$$\|f\|_{L^1} = \int_{-1}^1 |f(x)| dx; \quad \|f\|_{L^2} = \left(\int_{-1}^1 |f(x)|^2 dx \right)^{1/2}.$$

Let $\{f_j\}_{j=1}^\infty$ denote a sequence of functions in L^2 . Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and

$$|\|f_j\|_{L^2} - \sqrt{2}| \leq 2^{-j}.$$

Show that $\lim_{j \rightarrow \infty} f_j(x) = 1$ for almost every $x \in [-1, 1]$.

Hint: Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that f_n converges weakly to $f \in L^2$ if

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx \quad \text{for all } g \in L^2(\mathbb{R}).$$

Find a sequence of bounded, (Borel) measurable sets in \mathbb{R} whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0 \in L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

7. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$g(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1; \\ 1 & \text{if } x^2 + y^2 \geq 1. \end{cases}$$

Find the distribution $(\partial_x^2 + \partial_y^2)g$.

8. For any $m \in \{1, 2, \dots\}$ let $f_m(x) = |x|^{-m}$ on $\mathbb{R} \setminus \{0\}$.

Find a distribution T_m on \mathbb{R} which agrees with f_m on $\mathbb{R} \setminus \{0\}$, i.e.

$$T_m(\phi) = \int_{\mathbb{R}} f_m(x)\phi(x) dx \quad \text{for any } \phi \in C_0^\infty(\mathbb{R} \setminus \{0\}).$$

Give complete justifications (in particular show that your choice of T_m really defines a distribution).

9. Let H be an infinite dimensional Hilbert space.

(i) Prove that there is no relatively compact neighborhood of the origin.

(ii) Let $T : H \rightarrow H$ be linear, bounded and surjective, and let B be the closed unit ball centered at the origin. Show that $T(B)$ is not compact.