Qualifying Exam in Analysis
Real and Complex Analysis (Math 721-722) Version
August 2007

Instructions: *Do six of the nine problems. To receive credit on a problem. you must show your work and justify your conclusions. To facilitate grading, please use a separate packet of paper for each question. Use a black pen or #2 pencil (no mechanical pencils please!).*

1. Let f be a continuous function on $\mathbb{R}$ and let for $n = 1, 2, \dots,$

$$F_n(x) = \int_0^x (x-t)^{n-1} f(t)dt.$$

Prove that F_n is n times differentiable, and prove a simple formula for its n-th derivative.

2. Given a sequence $\{c_n\}_{n=0}^\infty$ of complex numbers, we let $s_n = \sum_{k=0}^n c_k$ denote the partial sums and $\sigma_N = \frac{s_0+\ldots+s_N}{N+1}$ their arithmetic means. We say that the series $\sum_{n\geq 0} c_n$ is *Cesáro summable* to σ if $\lim_{N\to\infty} \sigma_N = \sigma$.

Show that if $\sum_{n\geq 0} c_n$ is Cesáro summable to σ and $\lim_{n\to\infty} nc_n = 0$ then the series $\sum c_n$ converges and $\sum_{n=0}^\infty c_n = \sigma$. *Hint:* First write $\sigma_N - s_N = \sum \alpha_k c_k$ with suitable coefficients α_k.

3. Recall the following definition: a function $g : (0,1) \to \mathbb{R}$ has *bounded variation* if

$$\sup_{N\geq 2} \sup_{0<x_N<\ldots<x_1<1} |g(x_1) - g(x_2)| + \ldots + |g(x_{N-1}) - g(x_N)| < \infty,$$

where the second sup is taken over all strictly decreasing sequences $x_N < \cdots < x_1$ with $x_i \in (0,1)$.

Find the exponents p for which the function $f : (0,1) \to \mathbb{R}$,

$$f(x) = x^p \sin(1/x)$$

has bounded variation.

4. Consider the inequality

$$\Big| \int_{\mathbb{R}} \prod_{i=1}^n f_i(x)dx \Big| \leq \prod_{i=1}^n \Big(\int_{\mathbb{R}} |f_i(x)|^{p_i} dx \Big)^{1/p_i} \qquad (*)$$

for measurable functions on $\mathbb{R}$ and $p_i \in (1,\infty)$, $i = 1, \dots, n$.

(a) Assume that (*) holds for all measurable $f_1, \dots, f_n$. Prove that necessarily $\sum_{i=1}^n 1/p_i = 1$.

(b) Conversely, show that if $\sum_{i=1}^n 1/p_i = 1$ holds then (*) holds for all measurable $f_1, \dots, f_n$. (Note: The familiar case $n = 2$ can be assumed).

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5. (i) Prove the identity

$$\{y : y \in E_k \text{ for infinitely many } k\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.$$

(ii) Let A be the set of points $x \in (0,1)$ with the property that there are infinitely many fractions p/q with integers p, q such that

$$|x - p/q| < 1/q^3.$$

Show that A is a set of measure 0.

6. Assume that $I = [a,b]$ is a compact interval and $f \in L^2(I)$ (with the usual Lebesgue measure). Show that if

$$\int_{[a,b]} f(x) x^n dx = 0 \text{ for } n = 0, 1, 2, \ldots,$$

then $f(x) = 0$ almost everywhere in I.

7C. Evaluate the following improper integrals which involve a parameter $a > 0$.

(a) $\displaystyle \int_{-\infty}^{+\infty} \frac{x \sin(x)}{x^2 + a^2} \, dx;$

(b) $\displaystyle \int_0^{\infty} \frac{1}{x^2 + a^2} \frac{dx}{\sqrt{x}}.$

8C. For each of the following, either construct a holomorphic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ with the stated properties, or show that no such function exists.

(a) For every integer $n \geq 2$ we have $f\left(\frac{1}{n}\right) = \frac{1}{n-1}$ and $f\left(\frac{i}{n}\right) = \frac{1}{in - 1}$.

(b) For every integer $n \geq 1$ we have $\left|f^{(n)}(0)\right| \geq (n/3)^n$.

(c) The function f extends to a continuous function on the closure of $\mathbb{D}$, $\left|f(e^{i\theta})\right| = 1$ for $0 \leq \theta \leq 2\pi$, and $f\left(\frac{i}{2}\right) = f\left(\frac{1}{2}\right) = 0$.

9C. Let $\Omega \subset \mathbb{C}$ be an open set containing the point 0. Suppose that $f : \Omega \to \Omega$ is a holomorphic mapping, with $f(0) = 0$ and $f'(0) = 1$. Suppose that f has the Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ at 0. Define the iterates of $f_k : \Omega \to \Omega$ by setting $f_1(z) = f(z)$, $f_2(x) = f\big(f_1(z)\big)$, and then by induction, $f_k(z) = f\big(f_{k-1}(z)\big)$.

(a) Show that if $f(z) = z + a_n z^n + O(z^{n+1})$, then $f_k(z) = z + k a_n z^n + O(z^{n+1})$.

(b) Show that if Ω is bounded and connected, then $f(z) \equiv z$ for all $z \in \Omega$.