

# QUALIFYING EXAM

in

## ANALYSIS

Department of Mathematics

University of Wisconsin-Madison

Wednesday, August 24, 2005

Versions for Math 722

**Instructions:** Do six of the nine questions. To facilitate grading, please use a separate packet of paper for each question. To receive credit on a problem, you must show your work and justify your conclusions.

### Standard notation used on the Analysis exams:

- (1)  $\mathbb{R}$  and  $\mathbb{C}$  denote the fields of real and complex numbers respectively.
- (2)  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  denotes the unit disc in the complex plane.
- (3) For points  $x$  and  $y$  in  $\mathbb{R}^n$ ,  $|x - y|$  denotes the Euclidean distance between the points.
- (4) If  $E \subset \mathbb{R}^n$  is a Lebesgue measurable set, then  $|E|$  denotes its Lebesgue measure.
- (5) If  $\mu$  is a positive measure on a set  $X$ , and  $f$  is a complex valued measurable function on  $X$ , then for  $1 \leq p < +\infty$ ,

$$\|f\|_p = \left[ \int_X |f(x)|^p d\mu(x) \right]^{1/p}.$$

Two functions on  $X$  are said to be equivalent if they are equal except on a set of  $\mu$  measure zero. For  $1 \leq p < +\infty$ ,  $L^p(X) = L^p(X, d\mu)$  is the space of equivalence classes of complex valued measurable functions such that  $\|f\|_p < +\infty$ .

- (6) If  $\mu$  is a positive measure on a set  $X$ , and  $f$  is a complex valued measurable function on  $X$ , then

$$\|f\|_\infty = \inf \{t > 0 \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\}.$$

$L^\infty(X)$  is the space of equivalence classes of measurable, complex valued functions on  $X$  such that  $\|f\|_\infty < +\infty$ .

- (7)  $L^p_{\text{loc}}(\mathbb{R})$  is the space of measurable, complex valued functions on  $\mathbb{R}$  which belong to  $L^p(K)$  for every compact set  $K \subset \subset \mathbb{R}$ .

- (8) If  $f$  and  $g$  are measurable functions on  $\mathbb{R}$ , the convolution  $f * g$  is defined to be the function

$$f * g(x) = \int_{\mathbb{R}} f(x-t)g(t) dt$$

whenever the integral converges.

- (9) If  $T$  is a distribution and  $\varphi$  is a test function, then  $\langle T, \varphi \rangle$  denotes the value of the distribution applied to the test function.

*The Doctoral Exam Committee proofreads the qualifying exams as carefully as possible. Nevertheless, this exam may contain typographical errors. If you have any doubts about the interpretation of a problem, please consult with the proctor. If you are convinced that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In any case, never interpret a problem in such a way that it becomes trivial.*

**Problem I** Assume that  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  are sequences of nonnegative real numbers such that

- (i)  $a_n \leq a_{n+1}$  for any  $n = 1, 2, \dots$
  - (ii)  $b_n \geq b_{n+1}$  for any  $n = 1, 2, \dots$ , and  $\lim_{n \rightarrow \infty} b_n = 0$ .
  - (iii)  $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})$  is convergent.
- (a) Prove that  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .
- (b) Show that conclusion (a) may fail if assumption (i) is omitted.

**Problem II** Let

$$F(t) = \int_0^{\infty} e^{-x^2} \cos(tx) dx, \quad t \in \mathbb{R}.$$

- (a) Prove that  $F'(t) = -\frac{t}{2}F(t)$ . (Justify all steps.)
- (b) Find  $F(1)$ .

**Problem III** Prove that

$$xy + x^4 - y^4 = 0$$

admits a continuous solution  $y = f(x)$  for  $|x| < \frac{1}{100}$ .

**Problem IV** Let

$$F(x) = \int_{[0, \infty)} \frac{e^{-xt}}{|\sin t|^a} dt, \quad x > 0.$$

- (a) Find the values of the parameter  $a$  for which the function  $F$  is well-defined on  $(0, \infty)$  (i.e. the integrand is in  $L^1$ ).
- (b) Show that  $F \in C^\infty(0, \infty)$  for these values of  $a$ . (*Hint:* Use the Lebesgue Dominated Convergence Theorem.)

**Problem V** Assume that  $(\Omega, \Sigma, \mu)$  is a measure space and  $f \in L^p(\Omega)$  for some  $0 < p < \infty$ .

(a) Show that

$$(*) \quad \lim_{q \rightarrow \infty} \|f\|_{L^q} = \|f\|_{L^\infty}.$$

- (b) Does the conclusion (\*) still hold if we omit the assumption  $f \in L^p(\Omega)$  (proof or counterexample)?

**Problem VI** Construct a sequence of continuous functions  $f_n$  on  $[0, 1]$  such that  $0 \leq f_n \leq 1$ ,

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 0,$$

but the sequence  $f_n(x)$  does not converge for any  $x \in [0, 1]$ .

**Problem VII** Let  $\Delta$  be the unit disc in  $\mathbb{C}$ .

- (a) Find a sequence of holomorphic functions  $f_n: \Delta \rightarrow \mathbb{C} \setminus \{0\}$  such that  $f_n(0) = \frac{1}{2}$  and  $\lim_{n \rightarrow \infty} f_n(\frac{1}{2}) = 0$ .
- (b) Prove that if  $f: \Delta \rightarrow \Delta \setminus \{0\}$  is holomorphic and  $f(0) = \frac{1}{2}$ , then  $|f(\frac{1}{2})| > c > 0$  for some constant  $c$  independent of  $f$ .

**Problem VIII** Evaluate

$$\int_0^{\infty} \frac{\log x}{x^3 + 1} dx.$$

(Justify all steps.)

**Problem IX** Let  $f(z)$  be holomorphic on  $\operatorname{Re} z > 0$  and continuous on  $\operatorname{Re} z \geq 0$ . Assume that  $|f(z)| < e^{-|z|}$ . Show that  $f(z) \equiv 0$ . (Hint: Consider  $F(z) = f(z)e^{\frac{1}{2(1-\alpha)}z^\alpha}$  as  $\alpha$  tends to  $1^-$ .)